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Asset Allocation, Multivariate Position Based Trading, and the Stylized Facts

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I dedicate this work to my wife Katja and our children Anna and Joel.

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Bernd Pape

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Abstract

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The returns of virtually all actively traded financial assets share a set of common statistical characteristics, such as absence of serial correlations, a leptokurtic return distribution with power-law decay of extreme returns, and clustered volatility with different degrees of long-term dependence for varying powers of absolute returns. These empirical findings are so robust across various financial markets, that they have become known as so called stylized facts of financial returns in the econometrics literature.

Recently a body of literature has developed which attempts to explain these stylized facts with the interaction of a large number of heterogeneously behaving market participants, rather than postulating their existence already in an unobservable news arrival process, as is done in traditional finance. The present study contributes to this emergent literature on heterogeneous agent models in financial markets.

I take issue with a common assumption in the agent-based literature, that traders base their orders upon (risk adjusted) expected profits alone, that is in particular without taking their current portfolio holdings into account. It has been claimed earlier (Farmer & Joshi 2002) that such an assumption may imply unbounded portfolio holdings, which is economically hard to justify given alone the risk constraints that portofolio managers face.

Taking a prominent agent-based model (Lux & Marchesi 2000) as an example, I show that order based trading does indeed lead to unbounded positions and I explain why this must be the case. An alternative formulation is then suggested, which takes acquired portfolio holdings explicitly into account and implies bounded inventories. At the same time, the single risky asset model is extended into a multivariate framework containing a second risky asset and a riskfree bond. Asset allocation and security selection are modeled as seperate decision processes in line with common practice in financial institutions. The resulting dynamics are shown to replicate the stylized facts of financial returns in a similar vein as earlier agent-based models, but under more realistic assumptions regarding traders' behaviour and inventories.

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1 Introduction

This PhD thesis is devoted to offering a behavioural explanation of the stylized facts of financial returns in a multi asset market under realistic assumptions regarding both the investment behaviour of traders and their holdings. As such it belongs to the field of heterogeneous agent models, which attempt to explain statistical properties of financial time series endogenously with the interaction of only boundedly rational, heterogeneous market participants, rather than with exogeneous news processed by a perfectly rational representative agent alone.

Chapter 2 deals with the statistical properties of equity returns, most of which they share with financial returns in general. Absence of serial correlations, heavy tails, volatility clustering, long memory, multiscaling and a positive corellation between trading volume and return variance are common to returns of every acivively traded financial asset. This is why they have become known as so called *Stylized Facts*, which every viable statistical model of asset returns should be able to generate. Asymmetric effects such as the leverage effect, return anomalies, and details about the autocorrelation and moment structure of stock and stock index returns are more specific to equities and appear thus less central in such modelling efforts. In chapter 3 it will be demonstrated how difficult it actually is to come up with a viable model generating those stylized facts. The first model being capable of simultaneously generating at least the unvivariate stylized facts—the multifractal model of asset returns—has first been introduced in 1997.

In chapter 4 I shall turn to behavioural models that have been offered in order to explain the stylized facts of financial returns. Particular emphasis will be given to the model by Lux & Marchesi (2000), as the model I shall suggest in chapter 5 may well be regarded as a multivariate extension to their univariate setup. On the practical side it appears reasonable to first rebuild their model in order to cross-check for any technical or methodological errors, before programming my own specifications. The results of this pre-testing will also be a part of chapter 4.

In chapter 5 I shall then extend the univariate model by Lux & Marchesi (2000) with one risky asset into a multivariate setup containing a second risky asset and a riskless bond. In order to add some further realism to the model, the investment process will be split up into asset allocation and security selection, as is common practice in financial institutions (see e.g. Davis & Steil (2001)).

The interaction of chartists and fundamentalists on multiple assets has also been considered by Westerhoff (2004) who generates return series similar to those observed in real markets. My main contribution relative to his study and those by Lux & Marchesi (1999, 2000) consists in removing inconsistencies concerning traders inventories resulting from the order-based setup of their models. Both Lux and Westerhoff consider trading at disequilibrium prices in order-driven markets following the tradition initiated by Beja & Goldman (1980) and Day & Huang (1990). That is, traders place orders proportional to the expected profits of their investments, while a market maker adjusts prices proportional to net excess demand, filling any imbalances between demand and supply from his inventory. The consequences of such a setup upon traders inventories remained unexplored until Farmer & Joshi (2002) pointed out that pure order-based trading implies non-stationary positions and traders can accumulate unbounded inventories, which is unacceptable from a risk management point of view.

Order-based trading appears also unrealistic because it is well established standard in the academic literature at least since Markowitz (1959), that investors consider portfolio holdings rather than orders as the relevant object of profit and risk considerations. The inconsistencies of an order-based setup become particularly obvious when extending a univariate model into a multi asset framework. Suppose for example that a trader has favoured asset A over asset B for a while, but receives now a signal which favours asset B over asset A. A consistent model would require the trader to close or at least diminish his position in asset A before entering a new position in asset B. That is, a new signal favouring B over A would not only generate buying orders for B, but also selling orders for A, until the desired new positions in assets A and B are established. This is not achieved by naïvely extending the order-based setup by Beja & Goldman to multiple assets, as it would falsely neglect any acquired position in A when producing new orders for asset B.

The traders in my model use therefore position-based rather than order-based trading strategies. That is, they choose portfolio holdings (rather than producing orders) proportional to expected investment profits. Trading orders are generated only when target portfolios change, as is expressed by the derivatives of target holdings with respect to time. In chaper 5 I shall demonstrate that the neat duplications of real financial returns' statistical properties in Lux' model extend to both the index and single asset returns in a multivariate setup with two risky stocks and a riskless bond, when asset allocation and security selection are modeled as separate decision processes and traders use position-based rather than order-based strategies. Chapter 6 will conclude.

2 Statistical Properties of Stock Returns

2.1 Unit of Measurement

From the viewpoint of an investor, the relevant quantity to measure the performance of an investment at time t over an investment period τ is its *return* $R_t(\tau)$ defined as the appreciation of its market value V scaled by its original market value: $R_t(\tau) = (V_t - V_{t-\tau})/V_{t-\tau}$.

In sufficiently liquid markets we may assume the market price P to be independent of the quantity purchased or sold, such that the return of an investment in identical non dividend bearing assets may be written as

$$R_t(\tau) = \frac{P_t - P_{t-\tau}}{P_{t-\tau}} = \frac{P_t}{P_{t-\tau}} - 1. \tag{2.1}$$

The return of an investment in stocks may generally not be calculated by (2.1) above, since stocks as a rule pay dividends. Also capital adjustments such as stock splits and stock dividends imply changes in market prices which do not reflect corresponding changes in investment value.

Returns of dividend paying stocks may thus only be written in the form (2.1) if market prices are adjusted to neutralize the effects of dividend payments and capital adjustments. Such *adjusted prices* are nowadays provided by most data vendors and are the appropriate building blocks for the analysis of meaningful investment returns. As is common in the empirical finance literature, we will refer with P to the adjusted rather than the quoted market prices.

Returns depend upon the the investment horizon τ: Multiperiod returns are products of single period returns.[1] The calculation of multiperiod returns as products of single period returns complicates the analysis of returns over different investment horizons

[1]More precisely, the multiperiod return $R_{t+\tau}(\tau = \tau_1 + \tau_2 + \cdots + \tau_n)$ is related to the subperiod returns $R_{t+\sum_{i=1}^{j}\tau_i)}(\tau_j)$, $j = 1, \ldots, n$ by the following product: $(1 + R_{t+\tau}(\tau)) = (1 + R_{t+\tau_1}(\tau_1)) \cdot (1 + R_{t+\tau_1+\tau_2}(\tau_2)) \cdots (1 + R_{t+\sum_{i=1}^{n}\tau_i}(\tau_n))$.

somewhat. For example, if we assume single period returns independently identically distributed (iid) under some symmetric distribution, the corresponding multiperiod return will be increasingly right-skewed as a function of the investment horizon just due to the multiplication of single period returns.

From a statistical point of view, it is then desirable to transform returns in such a way that multiperiod returns may be constructed from sums rather than products of single period returns. Such a transformation is given by introducing *logreturns* $r_\tau(t)$ as

$$r_t(\tau) = \ln(1 + R_t(\tau)) = \ln P_t - \ln P_{t-\tau}. \tag{2.2}$$

Multiperiod returns over long investment horizons become then normally distributed for iid returns by virtue of the central limit theorem[2]. Logreturns are also called *continously compounded returns* because they represent the yield of an investment under continuous compounding. Their difference from simple returns remains negligible for returns in the range of $\pm 15\%$, implying that logreturns may be cross-sectionally aggregated with negligible loss of accuracy for investment horizons up to at least one week, as long as no extraordinary returns occur.

The use of returns (or logreturns) rather than (adjusted) prices in the analysis of financial time series may also be motivated statistically by the so called *unit root property* of asset prices and their logs. That is, in autoregressions of the Dickey-Fuller type

$$\ln P_t = \rho \ln P_{t-1} + u_t \tag{2.3}$$

with stationary increments u_t one is generally unable to reject the hypothesis $\rho = 1$,[3] implying difference stationarity of the differenced series as is obtained by taking logreturns. This provides an additional argument for the use of returns beyond that of reflecting the investors viewpoint, since stationary time series are easier to analyse than those having a unit root.

[2] provided that single period returns have finite variance. For a discussion of the general case, see section 3.2.1.

[3] see, for example, pages 18–21 in Pagan (1996).

2.2 Absence of Serial Correlation

The fact that stock price changes appear to be uncorrelated was already noted by King (1930). Kendall & Hill (1953) provide the first rigorous analysis of the time series of stock indices. They find only small (and usually positive) autocorrelations in the weekly return series of 19 British stock indices in 1928-38, half of them insignificant. Even the highest measured autocorrelation coefficients stay below 0.24, implying predictability (R^2) of less then 6% of a weeks return by the return of the preceding week.

Fama (1965) investigated in his doctoral thesis both daily and weekly returns of individual stocks in 1957-62. He found small predominantly positive autocorrelations (usually below 0.1) at daily and even smaller predominantly negative autocorrelations (usually above -0.05) at weekly frequency. A rapid decline of the autocorrelation above the first lag has since then be confirmed in many studies for both stocks and stock indices,[4] and even for high frequency data,[5] making absence of autocorrelations in returns a well accepted working hypothesis for all horizons despite its marginal rejection at the first lag.

2.3 Excess Kurtosis

Returns of stocks and stock indices, like the returns of many other financial assets, are bell shaped similar to the normal distribution, but contain more mass in the peak and the tail than the Gaussian. Such distributions are called *leptokurtic*. Leptokurtosis becomes visually evident as a curve shaped as an elongated S in so called QQ-plots, in which the quantiles of an empirical distribution are plotted against the corresponding quantiles of a normal distribution with mean and variance identical to those of the empirical distribution.

[4] see for example Fama (1970, 1976); Taylor (1986); Ding, Granger & Engle (1993) and references therein.

[5] see Gopikrishnan et al. (1999).

Osborne (1959) contains such plots with the characteristic elongated S shape of leptokurtic returns, but he did not comment on this obvious deviation from normality. First Alexander (1961) noted that Osborne's data appeared to contain far more large price changes than are characteristic of a normal distribution. Fama (1965) found leptokurtic returns in each of 30 constituents of the Dow Jones Industrial Average stock index and Mandelbrot (1963) references leptokurtosis in other financial time series back to 1915.

Leptokurtosis manifests itself mathematically in having a *kurtosis* (or coefficient of kurtosis) larger than 3, which is the kurtosis of the normal distribution. The coefficient of kurtosis κ of a random variable X is defined as

$$\kappa(X) = \frac{\mathbf{E}[X - \mathbf{E}(X)]^4}{\{\mathbf{E}[X - \mathbf{E}(X)]^2\}^2}, \tag{2.4}$$

where $\mathbf{E}(\cdot)$ stands for the mathematical expectations operator. Some studies define kurtosis as the difference between κ and its benchmark 3. The normal distribution would then have a kurtosis of 0. In this study we will call $\kappa - 3$ *Excess Kurtosis* and use the terms kurtosis and coefficient of kurtosis as synonyms, such that a normally distributed variable has a kurtosis κ of 3 and an excess kurtosis of 0. Studies with sampling frequency higher than 1 month report consistently kurtosis in excess of 3, often even 2 digit numbers, no matter whether investigating individual stocks or stock indices and independent of the time period and region considered.[6]

While the finding of excess kurtosis appears to be a robust result also for financial time series other than equities,[7] the finding of 2 digit numbers for κ has to be interpreted with care. Raising deviations from the mean in (2.4) to the 4th power implies that kurtosis estimates are highly sensitive to outliers. More robust measures of kurtosis tend to report still consistent but much milder excess kurtosis with less fluctuations over subperiods than the traditional measure κ.[8] Furthermore we shall see below, that κ need not necessarily be well defined for stock and stock index returns, which impedes its usefulness in the analysis of such time series.

[6] see for example Fama (1976); Schwert (1990); Campbell, Lo & MacKinlay (1997); Aparicio & Estrada (2001) and references therein.

[7] see for example Pagan (1996); Farmer (2000); Cont (2001) and references therein.

[8] see for example Kim & White (2004).

2.4 Heavy Tails

The kurtosis κ of a random variable X is a measure of its dispersion around the two values $\mu \pm \sigma$, where μ and σ stand for the expected value and standard deviation of X, respectively.[9] This implies that κ grows with probability mass both in the center and the tails, and declines with probability mass in the shoulders. For risk-mangagement purposes, however, it is desirable to have a measure of fat-tailedness only.

Extreme value theory[10] provides such a measure through its classification of the limiting distributions of sample extremes of iid random variables with continuous distributions. Denoting with $M_n = \max\{x_1, x_2, \ldots, x_n\}$ the maximum of n sample observations of the iid random variables $X_1, X_2, ..., X_n$, it has been shown by Fisher & Tippett (1928), that there exist only three classes of non-degenerate limiting distributions for suitably shifted and rescaled sample maxima M_n in the limit $n \to \infty$, called *Generalized Extreme Value* (GEV) distributions:

1. Gumbel (GEV Type I): $$G_I(x) = \exp\{-e^{-x}\}, \quad x \in \mathbb{R}, \tag{2.5}$$
2. Fréchet (GEV Type II): $$G_{II,\alpha}(x) = \exp\{-x^{-\alpha}\}\mathbb{I}_{x>0}, \tag{2.6}$$
3. Weibull (GEV Type II): $$G_{III,\alpha}(x) = \exp\{-(-x)^{\alpha}\}\mathbb{I}_{x\leq 0} + \mathbb{I}_{x>0}. \tag{2.7}$$

where $\mathbb{I}_{x>0}$ and $\mathbb{I}_{x\leq 0}$ denote the corresponding indicator functions and α is a positive shape parameter often denoted as *Tail Index* for reasons that will become apparent below. Their representation may be unified within the so called von Mises parametrization as

$$G_\xi(x) = \exp\{-(1 + \xi x)^{-1/\xi}\}, \tag{2.8}$$

where the sign of the shape parameter ξ determines the type of the limiting distribution: $\xi > 0$ for Fréchet (II), $\xi < 0$ for Weibull (III) and $\xi \to 0$ for Gumbel (I). ξ is related to α by $\xi = 1/\alpha$ in the type II (Fréchet) case and $\xi = -1/\alpha$ in the type III (Weibull) case.[11]

[9] see Moors (1988).

[10] Recent expositions of extreme value theory include Adler, Feldman & Taqqu (1998); Embrechts, Klüppelberg & Mikosch (1997) and Reiss & Thomas (1997).

[11] Some studies denote the parameter ξ rather than α as tail index. We shall use this term for the parameter α as it has the more intuitive interpretation as the highest defined moment of X_i in distributions with infinite support (see below).

The survival or tail probabilities $\bar{F}(x) = P(X > x)$ of a random variable X whose maxima are described by one of the GEV distributions, is connected to $G(x)$ through the relation:

$$\bar{F}(x) = -\ln G(x), \qquad \text{if } \ln G(x) > -1. \tag{2.9}$$

This implies the following tail probabilities for the random variables X:

Type I:	Medium-tailed	$\bar{F}(x) = \exp(-x)\mathbb{I}_{x\geq 0}$,	(2.10)
Type II:	Fat-tailed	$\bar{F}(x) = x^{-\alpha}\mathbb{I}_{x\geq 1}$,	(2.11)
Type III:	Thin-tailed	$\bar{F}(x) = (-x)^{\alpha}\mathbb{I}_{-1\leq x\leq 0}$.	(2.12)

The labels medium-, fat-, and thin-tailed in (2.10) to (2.12) refer to the decay of $\bar{F}(x)$. We see that random variables whose extremes may be described by Gumbel (type I) or Fréchet (type II) distributions are characterized by exponentially respectively hyperbolically declining tails, whereas distributions with extremal behavior of type III (Weibull) have finite endpoints. Any distribution with limiting extremal behavior may then be classified into one of the three types according to the asymptotical decay of its tails. Note that the tail index α coincides in the case of fat-tailed distributions (type II) with the exponent of the hyperbolic decay, implying non-existence of any moments higher than α for such distributions.

A unifying representation of (2.10) to (2.12) is given by the survival or tail probability of the *Generalized Pareto Distribution* (GDP):

$$\bar{F}_{\xi}(x) = (1 + \xi x)^{-1/\xi} \tag{2.13}$$

where the sign of ξ classifies the distribution into type I ($\xi \to 0$), type II ($\xi > 0$) and type III ($\xi < 0$), and the tail index α is related to ξ by the identity $\alpha = 1/|\xi|$ as in (2.8) above.

Hill (1975) provides the following maximum likelihood estimator for ξ conditional on the tail size:

$$\hat{\xi} = \frac{1}{k}\sum_{i=1}^{k}\{\ln x_{(n-i+1)} - \ln x_{(n-k)}\} \tag{2.14}$$

where $x_{(i)}$ denotes the i'th order statistics of the sample and k denotes the number of the n sample observations for which the asymptotic behavior described in (2.10) to (2.13) is assumed to be valid.

Jansen & de Vries (1991) apply the Hill-estimator to daily returns of ten US stocks and two stock indices in 1962–86 and obtain estimates for the tail index α in the range 3.2–5.2. Loretan & Phillips (1994) obtain α–estimtes in the range 3.1–3.8 for daily returns of the S&P 500 index in 1962–87 and 2.5–3.2 for monthly stock index return series from 1834–1987. Abhyankar, Copeland & Wong (1995) and Longin (1996) investigate a data set of daily US stock return series from 1985–1990 at various frequencies and find estimates for the tail index in the range 3–4. Lux (1996b) applies the Hill estimator to daily returns of the German share index DAX and its constituents in 1988–94 and obtains estimates for α in the range 2.3–3.8.

While the existence of the 4th moment (kurtosis) cannot decisivly be ruled out from the studies above, it appears at least questionable for return periods of 1 day and above. The existence of the 3rd moment (skewness) appears somewhat more likely, though not guaranteed, whereas the the consistent finding of tail index estimates significantly above 2 points towards the existence of the 2nd moment (variance) of the return generating process.

Estimates for the tail index α in high frequency returns below 1 day down to 1 minute yield values in a much closer range around 3,[12] where the existence of kurtosis can be definitely ruled out while the existence of skewness remains possible.

2.5 Heteroscedasticity and Volatility Clustering

The absence of autocorrelation discussed in section 2.2 does not rule out the presence of nonlinear dependencies between returns of stocks and stock indices. Indeed it has been found that tests for for independence like the BDS test by Brock, Dechert

[12] see Gopikrishnan et al. (1998, 1999) and Plerou et al. (1999).

& Scheinkman (1987) regularly reject the null hypothesis of independence of equity returns as of financial returns in general[13].

Even swift visual inspections of plots of equity return series as of financial return series in general reveal *Heteroscedasticity* as the most obvious violation of the assumption of independently and identically distributed returns: volatility as measured by absolute or squared returns is not constant through time.

Heteroscadasticity was first noted by Mandelbrot (1963) in daily returns of cotton prices. Fielitz (1971) investigated the returns of 200 stocks listed at the New York Stock Exhange (NYSE) from 1963-68 and found that almost half of the stocks investigated exhibited significant variation in realized volatility of the daily returns. For weekly returns the fraction with statistically significant heteroscedasticity was one quarter[14].

Schwert (1989) reports volatility estimates of monthly stock returns in 1857-1987 varying from 2% in the early 1960's to 20% in the early 1930's. Haugen, Talmor & Torous (1991) identify more than 400 significant changes in volatility of the daily price changes in the Dow Jones Index in 1887-1988.

Volatility is not only fluctuating but also correlated through time. Again this fact has first been noted by Mandelbrot for daily returns of cotton prices in his famous statement that

> large changes tend to be followed by large changes—of either sign—and small changes tend to be followed by small changes.
> (Mandelbrot 1963: page 418).

Fama (1965) finds an increased conditional probablility of large price changes on stocks with large price changes on the preceding day in a sample of 10 randomly selected US

[13]see Scheinkman & LeBaron (1989); Hsieh (1991); Brock, Hsieh & LeBaron (1991); Bollerslev, Engle & Nelson (1994); Pagan (1996).

[14]Further early illustrative examples of heteroscedasticity in equity returns include Wichern, Miller & Hsu (1976) and Hsu (1977, 1979a, 1982).

stocks.

Engle (1982) suggest a Lagrange Multiplier test to test the assumption of Gaussian white noise $\epsilon_t|I_{t-1} \sim \mathcal{N}(0, \sigma^2)$ in the dynamic regression model $y_t = x_t\beta + \epsilon_t$ against the time varying alternative

$$\epsilon_t|I_{t-1} \sim \mathcal{N}(0, \sigma_t^2), \quad \sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j \epsilon_{t-j}^2 \equiv \alpha_0 + \alpha(L)\epsilon_t^2 \tag{2.15}$$

where $\epsilon_t|I_{t-1}$ denotes the residuals conditional on the information set I_{t-1}, $\mathcal{N}(0, \sigma_t^2)$ denotes the normal distribution with mean 0 and time-varying variance σ_t^2, the α_j's are non-negative parameters not to be mixed up with the tail index, L is the back-shift operator and $\alpha(L)$ is the correlsponding polynomial in L with coefficients α_j. Engle named this alternative ARCH for *Auto**R*egressive *C*onditional *H*eteroscedasticity. ARCH effects have been been extensively documented for a wide range of financial time series, including stock and stock index returns[15].

ARCH effects provide a potential explanation for leptokursis of returns by application of Jensen's inequality to $(\sigma_t^2)^2$ in (2.15). Assuming the returns R_t to be ARCH-distributed implies for the standardized return

$$z_t \equiv \left.\frac{R_t}{\sigma_t}\right| I_{t-1} \sim \mathcal{N}(0, 1)$$

which yields for the kurtosis of the return process:

$$\frac{\mathbf{E}(R_t^4)}{\mathbf{E}(R_t^2)^2} = \frac{\mathbf{E}(z_t^4) \cdot \mathbf{E}(\sigma_t^4)}{\mathbf{E}(z_t^2)^2 \cdot \mathbf{E}(\sigma_t^2)^2} \geq \frac{\mathbf{E}(z_t^4)}{\mathbf{E}(z_t^2)^2} = 3 \tag{2.16}$$

Note that the reasoning above is not confined to ARCH but may be applied to any heteroscedastic volatility process. As such, heteroscedasticity will always increase kurtosis, no matter whether the underlying volatility process is specified as ARCH or not.

[15] see e.g. Bollerslev (1987); French, Schwert & Stambaugh (1987); Lamoureux & Lastrapes (1990); Koutmos, Lee & Theodossiou (1994) and the reviews in Bollerslev, Chou & Kroner (1992); Gouriéroux (1997) and Degiannakis & Xekalaki (2004).

2.6 Long Range Dependence

Taylor (1986) shows that the autocorrelation function (ACF) for squared residuals in Engle's ARCH(p) process (2.15) follow the same Yule-Walker equation as a corresponding AR(p)process, which implies an exponentially declining ACF of the squared residuals for lags longer than the highest lag in the ARCH specification.

Visual inspections of autocorrelograms for absolute and squared financial returns raise however doubts over such fast a decay. For example the autocorrelogram in Taylor (1986: p.55) of absolute and squared stock returns betweeen 1966 and 1976 stays significantly positive over all lags plotted up to 30 days. Ding et al. (1993) calculate sample ACF's for various powers between 1/8 and 3 of absolute daily returns of the S&P 500 index in 1929–91 and find significant positive values at least up to lag 100, the first negative autocorrelation coefficient usually occuring around lag 2500 corresponding to a time interval of approximately 10 years.

Such findings have led to the consensus that the autocorellation structure of absolute and squared returns is better described by hyperbolic rather than exponential decline[16]. Hyperbolic decline in the autocorrelation function is a defining property of *Long Memory* or *Long Range Dependence* (LRD), which for stationary processes X_t with finite mean and variance may be equivalently defined as follows [17]:

1. *There exists a real number* $a \in (0,1)$ *and a constant* $c_\rho > 0$ *such that the autocorrelation function* $\rho(k) = \mathbf{E}[(X_t - \mu)(X_{t-k} - \mu)]/\sigma^2$ *has the asymptotic behavior*

$$\lim_{k\to\infty} \rho(k)/[c_\rho k^{-a}] = 1. \tag{2.17}$$

2. *There exists a real number* $b \in (0,1)$ *and a constant* $c_f > 0$ *such that the spectral density* $f(\lambda) = \frac{\sigma^2}{2\pi}\sum_{k=-\infty}^{\infty} \rho(k)e^{ik\lambda}$ *has the asymptotic behavior*

$$\lim_{\lambda\to 0} f(\lambda)/[c_f|\lambda|^{-b}] = 1. \tag{2.18}$$

[16] see Mantegna & Stanley (2000); Cont (2001); Lux & Ausloos (2002) and references therein.
[17] see Beran (1994: Chapter 2).

The parameters a and b are related to a long memory parameter called *Hurst Exponent* H by the identities $a = 2-2H$ and $b = 2H-1$. The Hurst exponent of a long memory process is thus in the range $1/2 < H < 1$.

The variance $\mathbf{V}$ of the averaged process $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ of a long memory process with Hurst exponent H scales asymptotically as[18]

$$\lim_{n\to\infty} \frac{\mathbf{V}(\bar{X}_n)}{c_\gamma n^{2H-2}} = \frac{1}{H(2H-1)} \tag{2.19}$$

implying hyperbolic decay in the variance of the time-averaged process with the same exponent a as in the autocorrelation function (2.17).

Long Memory has traditionally been detected using the *Rescaled Range* (R/S) statistics Q_n invented by Hurst (1951) as the standardized range of the partial sum of the first l deviations of X_j from the sample mean $\bar{X}_n$:

$$Q_n(l) \equiv \frac{1}{s_n}\left[\max_{1\le l\le n}\sum_{t=1}^{l}(X_t-\bar{X}_n) - \min_{1\le l\le n}\sum_{t=1}^{l}(X_t-\bar{X}_n)\right] \tag{2.20}$$

$$\text{with standard deviation estimator} \quad s_n \equiv \left[\frac{1}{n}\sum_{t=1}^{n}(X_t-\bar{X}_n)^2\right]^{1/2} \tag{2.21}$$

The R/S statistics has been developed further by Lo (1991) who increased its robustness against the effects of short-range dependence by modifying the standardization in (2.20) and derived asymptotic sampling theory for the modified statistics.

A related approach is given by *Detrended Fluctuation Analysis* (DFA) introduced by Peng, Buldyrev, Havlin, Simons, Stanley & Goldberger (1994). DFA divides the full sequence of n cumlative sums $Y_t = \sum_{\tau=1}^{t} X_\tau$, $t = 1, 2, \ldots, n$ into n/l nonoverlapping boxes of length l, substracts the local trend—determined as the slope of a least-squares regression—within each box, and calculates a test statistics $F_n(l)$ as the average standard deviation about the resulting detrended walk. Both statistics $Q_n(l)$ and $F_n(l)$ are expected to scale with H as l^H for large values of l with $H > 1/2$ in the presence of long range dependence.

[18]Theorem 2.2 in Beran (1994), $c_\gamma > 0$ is a constant.

A particular class of processes capable of producing long range dependence are the *fractionally integrated autoregressive moving average* (ARFIMA) models independently introduced by Granger & Joyeux (1980) and Hosking (1981) with spectral density

$$f(\lambda) = \frac{\sigma^2}{2\pi}\left|1 - e^{-i\lambda}\right|^{-2d} = \frac{\sigma^2}{2\pi}\left(4\sin^2\frac{\lambda}{2}\right)^{-d}. \tag{2.22}$$

The *fractional differencing parameter* $d \in (-0.5, 0.5)$ has the same sign as the autocorrelations of the observations generated by (2.22) and is related to the Hurst exponent H by the identity

$$d = H - 1/2. \tag{2.23}$$

Fractionally integrated autoregressive moving average processes with $0 < d < 1/2$ generate therefore positively autocorrelated observations with long range dependence. The spectral representation of the ARFIMA model (2.22) motivated Geweke & Porter-Hudak (1983) to determine d from a log-log regression of the sample analogon $I(\lambda_j)$ to the spectral density $f(\lambda)$ in (2.18)

$$I(\lambda_j) = \frac{1}{2\pi n}\left|\sum_{t=1}^{n}(X_t - \bar{X})e^{it\lambda_j}\right|^2, \qquad \bar{X} = \frac{1}{n}\sum_{t=1}^{n}X_t, \tag{2.24}$$

evaluated at Fourier frequencies λ_j in finite samples of size n,

$$\lambda_j = \frac{2\pi j}{n}, \quad j = 1, 2, \ldots, (n-1)/2, \tag{2.25}$$

against the spectral density of an ARFIMA process (2.22),

$$\ln I(\lambda_j) = \beta_0 + \beta_1 \ln\left(4\sin^2\frac{\lambda_j}{2}\right) + \epsilon_j, \tag{2.26}$$

such that

$$\hat{d} = -\hat{\beta}_1, \qquad \hat{H} = \hat{d} + 1/2. \tag{2.27}$$

In long memory processes other than ARFIMA the spectral representation (2.22) may hold only approximately for small enough frequencies λ, such that the regression (2.26) of I upon λ_j is to be performed upon the lowest $m = g(n)$ Fourier frequencies only, with g usually chosen as $m = n^u$, where $u \approx 0.5$.

Identifying the presence of Long Range Dependence in squares of returns is important, since the slower than n^{-1} decline in variance (see (2.19)) may invalidate standard inferences about squared returns and volatility. In particular, the sample standard deviation defined in (2.21) applied to squared returns is biased with sampling variance decling slower than $1/n$, which implies errors in the ACF-estimates of squared returns, used for example in ARCH-modelling, with wider than expected confidence bands[19].

Furthermore, the slower than n^{-1} decline in autocorrelations (see (2.17)) implies that the infinite sum of autocorrelations is no longer finite, such that there exists no characteristic correlation time after which the process may be approximated as Markovian[20].

Crato & de Lima (1994) find long range dependence in the daily squared returns of 3 US stock indices in the time period from January 1980 to December 1990. Lobato & Savin (1996) extend this finding for absolute and squared returns of the S&P 500 index and the 30 constituents of the Dow Jones Industrial Average between July 1962 and December 1994. Lux (1996a) finds evidence for long memory in daily returns of the German share index DAX and its 30 constituents in 1959–88.

Long range dependence in high frequency equity returns has been reported for the US stock market e.g. by Cizeau et al. (1997); Liu et al. (1997, 1999) and for the Italian stock market by Raberto, Scalas, Cuniberti & Riani (1999)[21].

2.7 Multiscaling

When Ding et al. (1993) calculated the sample ACF as a function of various powers q of the absolute daily S&P 500 index returns ACF($|r|^q$),[22] they found that it was monotonically increasing for $q \lessapprox 1$ and monotonically decreasing for $q \gtrapprox 1$ independent of the time lag considered. This finding has been later confirmed for the same index by Pasquini & Serva (1999). Nonlinear scaling of the sample ACF in powers of q has also been reported for the German Dax index by Lux (1996a), for the British FT-SE

[19] see Beran (1994: Chapter 1) and the discussion in Mikosch (2003b).

[20] see the discussion in Mantegna & Stanley (2000).

[21] For evidence of long memory in financial time series of assets other than equities see the references in the review studies by Farmer (2000); Cont (2001) and Lux & Ausloos (2002).

[22] see section 2.6.

index by Mills (1997), and for the Spanish stock market by Grau-Carles (2000).

Such a non-linear scaling of absolute returns with their exponent fits well into the concept of *Multiscaling*, which Mandelbrot, Fisher & Calvet (1997) define as follows:

A stochastic process $\{X(t)\}$ is called multifractal *if it has stationary increments and satisfies:*

$$\mathbf{E}(|X(t)|^q) = c(q)t^{\tau(q)+1} \quad \forall t \in T, q \in Q \tag{2.28}$$

where T and Q are intervals on the real line with positive lengths, $0 \in T$, $[0,1] \subseteq Q$, and $\tau(q)$ and $c(q)$ are functions with domain Q. A multifractal process with nonlinear scaling function $\tau(q)$ is called multiscaling*, otherwise the process is called* uniscaling *or* unifractal (monofractal).

Mandelbrot et al. (1997) show that self-affine processes $\{X(t), t \geq 0\}$ satisfying $X(t) \stackrel{d}{=} t^H X(1)$[23] are unifractal with scaling function $\tau(q) = Hq - 1$. This suggests to define a generalized Hurst exponent H_q through the relation

$$\tau(q) = qH_q - 1. \tag{2.29}$$

The definition (2.28) above suggests to identify multiscaling by use of the sample analogon to $\mathbf{E}(|X(t)|^q)$, the so called *height-height correlation function of order* q or *q'th order structure function* defined by Barabási & Vicsek (1991) as

$$c_q(\Delta t) = \frac{1}{N} \sum_{i=1}^{N} |p(t_i + \Delta t) - p(t_i)|^q, \tag{2.30}$$

where $p(t_i)$, $i = 1, 2, ..., N$ denote the log-prices taken at N time points with equal distances Δt. If the log-prices p follow a multifractal process, the structure function c_q is according to (2.28) and (2.29) expected to scale with Δt as

$$c_q(\Delta t) \propto \Delta t^{\tau(q)+1} = \Delta t^{qH_q}. \tag{2.31}$$

[23] The sign $\stackrel{d}{=}$ denotes equality in distribution, here: $X(t)$ has the same distribution as $t^H X(1)$.

Multiscaling may then be identified by calculating the structure function c_q for various moments q, checking for power-law scaling in the time scale Δt to establish multifractality, and finally checking for non-linearity of the scaling function $\tau(q)$ to establish multiscaling.

The first application of this approach to financial returns has been by Vassilicos, Demos & Tata (1993) to find multifractality in the DM/$ exchange rate. Since then it has been applied e.g. to Gold, the DJIA stock index and the BGL/$ exchange rate by Ivanova & Ausloos (1999), to the German DAX index by Ausloos & Ivanova (2002), to 29 commodities and 2449 US stocks by Matia, Ashkenazy & Stanley (2003), and to 32 international stock indices, 29 foreign exchange rates and 28 fixed income instruments by Matteo, Aste & Dacorogna (2005), all of which find power-law scaling of the structure function c_q with nonlinear scaling function $\tau(q)$, which they interpret as evidence for multiscaling.

The scaling approach in (2.31) appears however somewhat limited in as much as monofractal processes may exhibit spurious multiscaling even in large finite data sets. An early example has been given by Berthelsen, Glazier & Raghavachari (1994), who show that finite samples of a monofractal random walk may exhibit spurious multiscaling over most of their scaling range. Veneziano, Moglen & Bras (1995); Bouchaud, Potters & Meyer (2000) and LeBaron (2001) provide further examples of spurious multiscaling. As such we cannot tell from finite data sets, whether the underlying stochastic process is truly multiscaling or not.[24]

2.8 Return Volume Relations

The academic treatment of the relationship between trading volume and stock returns goes back to Osborne (1959), who notes that

> volume tends to be larger when the market as a whole (i.e. all stock prices)

[24]See also the discussion in Lux (2001).

> heaves up or down most rapidly.
> (Osborne 1959: page 167).

Ying (1966) compared six years long time series of S&P500 prices and NYSE trading volume. He concluded among other findings that small (large) trading volumes were usually accompanied by falling (rising) prices, and that large volume increases were usually accompanied by either a large rise or a large fall in price. Ying was thus the first to report a positive correlation between trading volume and price change as well as its variance.

While Ying's work has been critized for methodological errors[25], the empirical findings themselves have been confirmed in later studies[26]. The empirical support appears to be somewhat stronger for the correlation between trading volume and price variance, which has also been reported for many time series of financial assets other than equities, than for the correlation between trading volume and returns themselves, which appears to have been reported for stocks and bonds only[27].

Tauchen & Pitts (1983) delvelope a microscopic model of sequential trading, which results in a joint mixture of independent normal distributions for both the price change and trading volume with the unobservable number of daily information events as the mixing variable, known as the bivariate *mixture of distributions hypothesis* (MDH). The MDH is attractive in as much as it has an economically meaningful interpretation of news affecting both prices and volume. It is furthermore consistent with a positive relationship between trading volume and return variance, as well as the empirically observed leptokurtosis of returns and positive skewness in the distribution of trading volume itself[28].

The MDH by Tauchen & Pitts (1983) has however not gone unchallenged. Richardson & Smith (1994) use the Generalized Methods of Moments (GMM) procedure by

[25]see e.g. Epps (1975); Karpoff (1987).

[26]see e.g. Epps & Epps (1976); Morgan (1976); Westerfield (1977); Rogalski (1978); Schwert (1989); Gallant, Rossi & Tauchen (1992) and other studies surveyed in Karpoff (1987).

[27]see Karpoff (1987).

[28]see Harris (1986).

Hansen (1982) to check whether the unconditional moments of price changes and trading volume conform with those of the mixture of distributions hypothesis. They reject MDH for all tested distributions of information flow basd upon the returns and trading volume of the 30 DJIA constituents in 1982–86. Jung & Liesenfeld (1996) arrive at similar conclusions based upon German stock market data in the time period 1990–94.

Andersen (1996) however, suggests a modified version of MDH based on a heterogeneous agent setting with asymmetrical information, resulting in Poisson-distributed trading volume conditional on the unobservable number of information events, which is not rejected by GMM. Liesenfeld (1998) generalizes the MDH setup by allowing for serial correlation in the mixing information variable, which had been assumed independent in both Tauchen & Pitts (1983) and Andersen (1996), but finds this insufficient to fully account for the empirically observed persistence in stock return variances.

A debate followed discussing in how much the MDH can account for long memory in the variance of the return process. Bollerslev & Jubinski (1999) find the same order of fractional integration from the hyperbolic decay in the ACF's of both trading volume and absolute returns of the S&P100 constituents in the time period 1962–95. They interpret this as evidence for a bivariate MDH specification, in which the latent information-arrival process has long memory. Also Lobato & Velasco (2000) find identical long-memory parameters in the returns and trading volume for most of the 30 DJIA constituents, but no evidence that both the return and the volume process are driven by the same long-memory component. Regúlez & Zarraga (2002) on the contrary, find evidence for a common latent factor driving both returns and trading volume in the Spanish stock market.

In judging these and similar studies one should keep in mind that trading volume, after all, might not be the best dimension to measure the impact of the unobservable information flow. For example Easley & O'Hara (1992) build a microstructure model, in which the time between trades rather than trading volume itself provides the most valuable information to market participants; and Ané & Geman (2000) show empirically that in order to recover a normal distribution for the high frequency returns of two technology stocks, time has to be rescaled with the the number of transactions rather than the volume of trading. Their view has however been recently challenged by Farmer et al. (2004) and Gillemot, Farmer & Lillo (2006) who find the price impact of individual market orders to be essentially independent of both trading volume and transaction frequency. Instead, they attribute both heavy tails, volatility clustering, and long memory to microstructure liquidity effects as measured by the distribution of gaps in the limit order book.

2.9 Asymmetric Effects

The positive return volume relationship discussed in section 2.8 in the sense that a time series responds differently to positive and negative shocks in the same or a related time series. Further examples of asymmetric effects in equity time series include the so called leverage effect and correlation breakdown, shortly to be discussed below.

2.9.1 Leverage Effect

A number of studies starting with Black (1976) report a negative contemporaneous relationship between volatility changes and returns at both stock and index level[29], commonly denoted as *Leverage Effect*. The term refers to a hypothesis by Black, that the volatility increase after price declines is due to the increased risk of the firm's equity as a result of its lower equity-to-debt ratio following negative returns.

Christie (1982) and Schwert (1989) test the leverage hypothesis and find qualitative support for it, although the elasticity of volatility changes with respect to financial levarge appears to be too small to take full account of the empirical observation. The latter finding has been recently confirmed by Figlewski & Wang (2000).

[29] see for example the studies by Christie (1982); French et al. (1987); Schwert (1989); Haugen et al. (1991); Campbell & Hentschel (1992); Cheung & Ng (1992); Gallant, Rossi & Tauchen (1993); Glosten, Jaganathan & Runkle (1993); Braun, Nelson & Sunier (1995); Duffee (1995); Tauchen, Zhang & Liu (1996) and Figlewski & Wang (2000).

The leverage hypothesis seems also an unlikely explanation since Engle & Lee (1993) found the asymmetric volatility response to stock price changes to be a transitory effect only. For example, Gallant et al. (1993) find that the leverage effect becomes insignificant after 5–6 days at index level and Tauchen et al. (1996) find a similar decline at individual stock level already after 2–3 days[30]. But firms are unlikely to adjust their capital structure that fast to the original level of financial leverage[31]. Also, if financial leverage was the true explanation for volatility asymmetry, then issue of debt and stock should be associated with a corresponding leverage effect as well; this has however not been found[32].

A competing explanation for the leverage effect is the so called *Volatility Feedback* hypothesis, according to which an increase in stock market volatility raises required stock returns, and thus lowers stock prices. It has also originally been proposed by Black (1976)[33] and termed such and empirically tested by Campbell & Hentschel (1992), who however find that volatility feedback has only little effect on returns.

Volatility feedback is also rejected in the studies by Bouchaud & Potters (2001) and Bouchaud et al. (2001) on high frequency returns, which find a negative correlation only between past returns and future volatility, but not the other way round. Bouchaud et al. (2001) manage to explain the leverage effect for individual stocks within a "retarded volatility" model in which price innovations at intraday frequency are assumed to be proportional to a moving average of past prices rather than the most recent price; but the explanation of the leverage effect at the index level requires the ad-hoc introduction of an additional "market panic" factor, whose existence remains theoretically unmotivated in their study.

As such, the economic mechanism behind the leverage effect remains an unsolved issue.

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[30] Exponential dampening of the leverage effect with slower decay for indexes than for individual stocks has been recently confirmed even for high frequency data, see Bouchaud & Potters (2001); Bouchaud, Matacz & Potters (2001); Litvinova (2003). They also confirm a finding originally noted by Braun et al. (1995), that the magnitude of the leverage effect appears to be stronger at market than at individual stock level.

[31] For related findings regarding adjustment of the capital structure to earnings-induced leverage variations, see Ball, Lev & Watts (1976).

[32] see Figlewski & Wang (2000).

[33] similar ideas are expressed e.g. in Malkiel (1979) and Pindyck (1984).

One should keep in mind, however, that the leverage effect appears to be small in size despite its statistical significance[34]. Furthermore it appears for the most extreme price movements only and is further attenuated by conditioning on trading volume[35]. In the light of such findings one might well be tempted to ask, whether there is much economical significance to the leverage effect at all[36].

2.9.2 Correlation Breakdown

Several studies find an increase of cross-correlations between equity returns in bear markets, which is commonly refered to as *Correlation Breakdown.* For example, King & Wadhwani (1990) and Lee & Kim (1993) find a significant increase in cross-correlations between the returns of several major stock indices after the October 1987 stock market crash. Erb, Harvey & Viskanta (1994) report higher correlations between the stock market returns of the G7-countries during recessions than in growth periods. A related effect is the increase of cross-market correlations during periods of high volatility as originally noted by Erb et al. (1994) and Longin & Solnik (1995) and recently confirmed by Ang & Bekaert (2002) and Das & Uppal (2004)[37].

Early studies suffered, however, from a flawed interpretation of correlation matrices conditioned on large versus small absolute ex post returns: Boyer, Gibson & Loretan (1999) show that correlations conditioned on threshold returns in only one of the series are biased upwards. Forbes & Rigobon (2002) use this insight to show that correlation breakdowns observed during the 1987 Stock Market Crash and other crises were only spurious, that is consistent with a constant unconditional correlation matrix between stock market returns. Loretan & English (2000) arrive at similar conclusions after investigating among others correlation breakdowns between the British FTSE-100 index and the German DAX index in the time period 1991–99.

[34] see Tauchen et al. (1996) and Andersen, Bollerslev, Diebold & Ebens (2001).

[35] see Gallant et al. (1992, 1993).

[36] For example, Bouchaud et al. (2001) deny such significance.

[37] The correlation increases during bear and volatilte markets are linked by the leverage effect, since the largest market moves tend to be declines, see e.g. Chen, Hong & Stein (2001).

In order to aviod spurious relationships between correlations and volatility or market trend, Longin & Solnik (2001) introduce the exceedence correlation function $\rho_{ij}^{\pm}(\theta)$ between normalized centered returns r_i, r_j above/below threshold θ as

$$\rho_{ij}^{\pm}(\theta) = \frac{\overline{r_i r_j}_{\gtrless\theta} - \overline{r_i}_{\gtrless\theta}\overline{r_j}_{\gtrless\theta}}{\sqrt{\left(\overline{r_i^2}_{\gtrless\theta} - \overline{r_i}_{\gtrless\theta}^{\,2}\right)\left(\overline{r_j^2}_{\gtrless\theta} - \overline{r_j}_{\gtrless\theta}^{\,2}\right)}} \tag{2.32}$$

where the subscript $\gtrless \theta$ means that both returns are larger than θ (resp. smaller than θ) for positive exceedence correlations $\rho_{ij}^{+}(\theta)$ (resp. negative exceedence correlations $\rho_{ij}^{-}(\theta)$) and the bar indicates the corresponding sample averages.

If asset returns were normal, the exceedence correlation function should asymptotically approach zero for both positive and negative thresholds. Longin & Solnik (2001) plot the exceedence correlation function for the monthly returns of several major stock indices in 1959–96 and find a decrease for positive θ only, but an increase with the absolute threshold for negative returns, indicating that cross-market correlations increase in bear markets, but not in bull markets. Similar results have been found by Ang & Bekaert (2002).

Turning to subportfolios and individual stocks, Ang & Chen (2002) find higher exceedence correlations between the aggregate US stock market and several style sorted subportfolios in bear than in bull markets for daily returns in 1963–98, and Bouchaud & Potters (2001) find the same pattern for daily returns for 437 S&P500 index constituents in 1990–2000[38].

Das & Uppal (2004) model correlation breakdown within a multivariate jump-diffusion process, where jumps occur simultaneously but their size is allowed to vary across assets. The idea is related to the non-Gaussian one-factor model by Bouchaud & Potters (2001)[39], where the individual stock return is modelled as a product of the retarded price and the sum of both market and ideosyncratic shocks. Ang & Bekaert (2002) however, claim the superiority of regime-switching models in explaining the observed difference between positive and negative exceedence correlations over both

[38] Cizeau, Potters & Bouchaud (2001) report similiar results for the daily returns of 450 US stocks in 1993–99.

[39] see section 2.9.1.

asymmetric GARCH and multivariate jump-diffusion processes.

2.10 Anomalies

2.10.1 Cross-Sectional Predictability

Although stock returns are sereially close to uncorrelated,[40] it appears cross-sectionally that stocks with certain characteristics offer higher returns than others even after controlling for risk. Such effects are called *anomalies* because investors should be indifferent about any characteristic of their investment other than its return and the risk associated with it. In how much the term "anomaly" is justified, depends then upon the quality of risk adjustment.

The predominant form of risk-adjusting stock returns is the deduction of expected returns from the *Capital Asset Pricing Model* (CAPM) by Sharpe (1964) and Lintner (1965a,b), which accounts for covariance risk with the market portfolio of all stocks, but ignores all other sources of risk; in particular intertemporal effects such as risk differentials in different stages of the business cycle or microstructure effects such as liquidity. Characteristics giving rise to a cross-sectional anomaly may also often be argued to be a proxy for expected returns.

The first cross-sectional anomaly was discovered by Nicholson (1968), who found that stocks with a low price earnings (P/E) ratio tend to outperform high P/E stocks. Basu (1977) showed on 1400 stocks traded on the New York Stock Exchange (NYSE), that the P/E effect survives risk adjustment by the CAPM: Buying the lowest P/E quintile and short-selling the highest P/E quintile would have generated 6.75% average abnormal return before trading costs in the period 1957–71.

Banz (1981) found that the 50 smallest NYSE stocks, measured in terms of market capitalization, outperformed the largest 50 NYSE stocks in 1931–75 by 1% per month

[40] see section 2.2.

on a risk-adjusted basis. Rosenberg, Reid & Lanstein (1985) found that stocks with a low price-to-book (P/B) ratio outperform high P/B stocks in a universe of 1400 highly capitalized stocks in the period 1973–84. All three effects (P/E, Size and P/B) have since then be confirmed by numerous further studies[41].

Fama and French argue in a series of papers [42], that size, price ratios such as price-to-book, price-to-earnings, dividend yield, price-to-cashflow and past sales growth rates may be subsumed in two additional risk factors to the CAPM for size and value. In how much the value effect is indeed a compensation for risk, or rather the result of psychologically biased, irrational investment decisions, is still a matter of intense debate between the above mentioned authors[43] and protagonists from the Behavioral Finance literature on the other side[44].

Another cross-sectional anomaly is the momentum effect discovered by Jegadeesh & Titman (1993), who find that stocks with above average returns over the last half year tend to outerperform over the following 3 to 12 months as well, consistent with delayed price reaction to firm specific news. The momentum effect has been confirmed e.g. by Chan, Jegadeesh & Lakonishok (1996); Brennan, Chordia & Subrahmanyam (1998); Fama (1998).

Cross-sectional anomalies have been aspersed of data-snooping e.g. by Lo & MacKinlay (1990); Black (1993); Breen & Korajczyk (1995); Kothari, Shanken & Sloan (1995); MacKinlay (1995). However, this appears to be an unlikely explanation, since the anomalies mentioned above have been frequently confirmed out of sample[45].

Brennan et al. (1998) argue that the size effect is indeed a liquidity effect, as the size factor in explaining abnormal returns is not robust to the inclusion of trading volume as an additional explanatory variable. Anyway there appears to be a consensus that the

[41] see e.g. the survey studies by Ziemba (1994) and Hawanini & Keim (1995).

[42] see Fama & French (1992, 1993, 1996).

[43] see also Fama & French (1995, 1998) and Fama (1998).

[44] see e.g. De Bondt & Thaler (1985); Chopra, Lakonishok & Ritter (1992); Lakonishok, Shleifer & Vishny (1994); Haugen & Baker (1996).

[45] see e.g. Hawanini & Keim (1995); Haugen & Baker (1996); Fama & French (1998); Rouwenhorst (1998); Davis, Fama & French (2000); Martikainen (2000).

size effect has largely disappeared since its publication in 1981, whereas the momentum effect seems to persist[46]. The persistence of the value effect is less clear. Schwert (2003) argues that the value effect has attenuated as well, whereas Hogan et al. (2004) demonstrates the profitability of several value-based strategies up until 2000, extending the sample period 1963–1990 originally used by Lakonishok et al. (1994)[47].

The most recent cross-sectional anomaly concerns the investment recommendations from brokerage analysts. Womack (1996) and Barber, Lehavy, McNichols & Trueman (2001) find that stocks with fresh buy recommendations outperform stocks with fresh sell recommendations, most probably reflecting finite reaction time to firm specific information. They report however excess returns, which are not sufficient to cover transaction costs.

2.10.2 Seasonal Anomalies

Seasonal Anomalies or *Calendar Effects* denote the empirical finding that stock returns appear not to be uniformly distributed over the year. The best known calendar effects include the *Weekend*, the *January*, the *Turn-of-Month* and the *Holiday Effect*.

Cross (1973) and French (1980) find unusually low returns on Fridays and extraordinary large returns on Mondays. Rozeff & Kinney (1976) find above average returns in January, which Keim (1983) and Reinganum (1983) show to be concentrated on firms with small market capitalization. Ariel (1987) and Lakonishok & Smidt (1988) find larger returns around the turn of the month; and Ariel (1990), Lakonishok & Smidt (1988) as well as Ziemba (1991) find extradordinary large returns on the days preceeding public hodlidays.

Seasonal anomamlies have been adversed of data-snooping—just like their cross-sectional counterparts discussed in section 2.10.1—due to the lack of any a priori theoretical ex-

[46]see Schwert (2003); Hogan, Jarrow, Teo & Warachka (2004).

[47]Hogan et al. (2004) do however not report isolated performance in the out-of-sample period 1991-2000.

planations for them. Calendar effect have been explained by institutional factors such as cash-flow and policy constraints and individual trading patterns such as tax-loss-selling and delayed reactions to market information[48], but all of these may just as well be regarded as *after the fact rationalizations* of empirically observed phenomena.

Seasonal anomalies are not stable through time just like their cross-sectional counterparts[49] and Sullivan et al. (1998) demonstrate that even the best performing calendar rules may be attenuated up to insignificance when correcting for data-snooping bias. Data-snooping alone, however, appears to be an insufficient explanation given their wide occurence in markets all over the world[50].

Schwert (2003) finds the January Effect to be confined to the cheapest and least liquid stocks, while the Weekend Effect seems to have disappeard since the early 1980's, suggesting that the market learns through time. The latter view is consistent with Bossaerts & Hillion (1999), who confirm in-sample predictability of international stock returns, but find no out-of-sample predictability even of the best in-sample models selected by standard statistical model selection criteria; and explain this with model nonstationarity due to learning by market participants.

[48] see Ziemba (1994).
[49] see Ziemba (1994) and Sullivan et al. (1998).
[50] see Ziemba (1994) and Hawanini & Keim (1995).

3 The Search for the Return Generating Process: Statistical Approach

3.1 Random Walk and Martingale Hypothesis

The earliest contribution to the theoretical description of return generating processes goes back to Bachelier (1900). Bachelier showed that an asset subject to independent price shocks must follow a Wiener process, which implies that a stock price measured at discrete time steps must follow a *Random Walk*:

$$P_t = P_{t-1} + \epsilon_t \tag{3.1}$$

with independently and identically distributed increments ϵ_t, which—according to Bacheliers analysis—should follow a normal distribution with zero mean and constant variance. We shall follow the most common usage of the term "Random Walk" in the literature by denoting with it any process following (3.1) with iid increments, thereby allowing the error term to follow other distributions as well.

Osborne (1959) uses psychological considerations to argue that the assumption of independence should apply to the logreturns defined in equation (2.2) of page 13, rather than to arithmetic price changes as in (3.1). The result has become known as *Geometric Brownian Motion* (GBM), in which logreturns are normally distributed, whereas gross returns $P_{t+\tau}/P_t = 1 + R_t(\tau)$ become lognormally distributed with probability density

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2\tau x}} \exp\left[-\frac{1}{2\sigma^2\tau}(\ln x - \mu\tau)^2\right], \tag{3.2}$$

where $\mu\tau$ and $\sigma^2\tau$ are the mean and variance of the normally distributed logreturns $r_t(\tau)$ over the investment period τ up to time t.

The random walk model squares well with absence of serial correlation in stock prices but is incompatible with heteroscedasticity and volatility clustering, since the price increments ϵ_t in (3.1) are assumed to be identically distributed, in particular, they have constant variance. This led Mandelbrot (1966) to introduce the *Martingale Model* of speculative prices, which assumes only $\mathbf{E}(\epsilon_t) = 0$ rather than requiring ϵ_t to be drawn from a fixed distribution.

Samuelson (1965, 1973) shows that the martingale model for stock prices discounted at the risk-free rate is consistent with an arbitrage-free market, which prices the stock at *Fundamental Value*, that is the expected present value of all future dividends. The latter hypothesis, that prices evolve as if market participants used the true probability distribution of events in making their predictions, has first been advanced by Muth (1961) and denoted by him as *Rational Expectations*.

3.2 Modelling the Unconditional Return Distribution

3.2.1 Infinite Variance Hypothesis

Significant excess kurtosis as well as strong time variation in variance of returns led Mandelbrot (1963) to argue for the use of *Lévy Stable Distributions* in the description of financial returns. The general class of Lévy stable distributions introduced by Lévy (1925) lacks any closed form solution, but may be describted by its characteristic function $\varphi_X(u) = \mathbf{E}(e^{iXu})$ as:

$$\ln \varphi_X(u) = \begin{cases} i\delta u - \gamma|u|^{\alpha_S}\left[1 - i\beta\frac{u}{|u|}\tan\left(\frac{\pi}{2}\alpha_S\right)\right] & \text{if } \alpha_S \neq 1, \\ i\delta u - \gamma|u|\left[1 + i\beta\frac{2}{\pi}\frac{u}{|u|}\ln|u|\right] & \text{if } \alpha_S = 1, \end{cases} \tag{3.3}$$

with location parameter $\delta \in (-\infty, \infty)$, skewness index $\beta \in (-\infty, \infty)$, scale parameter $\gamma \in (0, \infty)$ determining the width, and characteristic exponent $\alpha_S \in (0, 2]$ determining the shape of the distribution. The normal distribution corresponds to the special case $\alpha_S = 2$. In most other cases, the distribution function and density of X can only be obtained by numerically evaluating the inverse Fourier transform of (3.3).

All non-normal Lévy stable distributions are leptokurtic and have hyperbolically declining tails with tail index α identical to their characteristic exponent α_S, which implies that they have infinite variance (see section 2.4). Apparent variation in the variance of returns would then not imply true variation in the distribution of ϵ_t through time, but instead be a mere result of sampling error, as the law of large numbers would then be no longer applicable to 2nd nor any higher moments.

The stable distribution hypothesis explains also leptokurtosis at varying time horizons, since application of the central limit theorem—used in the derivation of normally distributed multiperiod returns—requires finite variance of the single period returns. Sums of independent increments with infinite variance, on the other hand, converge in distribtution to the non-normal members of the stable distribtution family.

As such, Lévy stable distributions are the only possible limiting distributions of independenly and identically distributed random variables[51]. Regarding long-term logreturns as sums of iid short-term logreturns, this would imply that Lévy stable distributions are the only candidates for describing long-term returns. DuMouchel (1973) showed, however, that the rate of convergence to the Lévy stable limit can be extremely slow in the case of infinite variances, requiring numbers of observations of order 10^3 before convergence to a stable limit could be observed. Furthermore, if returns are not identically distributed, then every *infinitely divisible* distribution, that is every distribution whose characteristic function φ may be expressed as the k'th power of some characteristic function φ_k: $\varphi(u) = [\varphi_k(u)]^k$, $k \in \mathbb{N}$, is permissable as a limit distribution for the sum of independent random variables[52].

Sums of iid stable random variables are themselves Lévy stable distributed with rescaled location and scale parameters, but identical skewness index β and characteristic exponent α_S as the individual summands. Regarding long-term logreturns as sums of iid short-term logreturns would then imply that long-term returns should have the same tail index α as their subperiod returns.

Some researchers[53] accepted Mandelbrot's infinite variance hypothesis merely upon indication of $\alpha < 2$ in their datasets without further testing of fit. Stability of the tail

[51] see Lévy (1925) and the discussions e.g. in Mandelbrot (1963) and Fama (1963).

[52] The result is due to Khintchine (1937). See also the discussion in Mantegna & Stanley (2000: page 30–33).

[53] see e.g. the studies by Fama (1965); Teichmoeller (1971); Simkowitz & Beedles (1980).

index α is however overwhelmingly rejected in a large number of studies starting with Officer (1972)[54], which find nonstationarity of α at different levels of time aggregation; in particular convergence to the normal distribution for longer time horizons, which implies the applicability of the central limit theorem, and thus finite variance. Further evidence against the infinite variance hypothesis is provided by numerous studies finding a tail index significantly larger than 2 [55], the empirically observed convergence of sample variance to finite values[56], and slower divergence of higher moments, than would be expected in the non-normal Lévy stable regime[57].

3.2.2 Combinations of Jump and Diffusion Processes

One possible avenue to generate leptokurtic returns without having to introduce infinite variance is to combine jump and diffusion processes. Press (1967) pioneered this approach by suggesting a compound Poisson process of normally distributed price increments as follows:

$$Z(t) \equiv \ln P_t = \ln P_0 + \sum_{k=1}^{N(t)} Y_k + X(t), \tag{3.4}$$

where $N(t)$ denotes a Poisson counting process representing the random number of information events, Y_k, $k = 1, \dots N(t)$, are normally distributed random variables representing the price reaction to such events, and $X(t)$ is an additional Wiener process to represent random price variation unrelated to information. All processes, $N(t)$, Y_k, and $X(t)$ are assumed to be mutually independent. Leptokurtosis is then introduced into this *Compound Events Model* by the Poisson mixture of normals.

Merton (1976) adds an extra drift term to (3.4)[58] and reinterprets the noise term $X(t)$ as ordinary price movements and the information induced price reaction Y_k as extraordinary jumps in order to obtain for the logreturn $r_\tau(t)$ defined in (2.2) on page

[54]further examples include Barnea & Downes (1973); Blattberg & Gonedes (1974); Hsu, Miller & Wichern (1974); Hagerman (1978); Upton & Shannon (1979); Fielitz & Rozelle (1983); Perry (1983).

[55]see section 2.4.

[56]see Cont (2001) and reference [22] therein.

[57]see Lau, Lau & Wingender (1990).

[58]Both the price reactions Y_k and the Wiener process $X(t)$ in Press' compound events model have zero mean.

13:

$$r_\tau(t) = \left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma\left(B(t+\tau) - B(t)\right) + \sum_{n=N(t+1)}^{N(t+\tau)} J_n \tag{3.5}$$

where $B(\cdot)$ denotes standard Brownian Motion and $N(\cdot)$ denotes a Poisson counting process with parameter λ; μ and σ denote drift and variance from the Brownian motion part of the process, respectively; and $J_n \sim \mathcal{N}(\mu_J, \sigma_J^2)$ represent the price change at the n'th jump. In this interpretation the model is commonly denoted as *Mixed Diffusion Jump* or simply *Jump Diffusion* process. Its probability density is given by

$$f(x) = \sum_{n=0}^{\infty} e^{-\lambda\tau}\frac{\lambda\tau^n}{n!}\phi\left(\mu\tau + n\mu_J, (\sigma\tau)^2 + n\sigma_J^2\right) \tag{3.6}$$

where $\phi(\mu, \sigma^2)$ denotes the probability density function of the normal distribution with mean μ and variance σ^2. As is apparent from the density function above, the jump diffusion process may just as well be regarded as a mixture of normals with infinitely many addends, leading to the denotation *Compound Normal*, although some authors reserve this term for discrete mixtures of normals with finitely many addends only (see below).

Oldfield, Rogalski & Jarrow (1977) and Oldfield & Rogalski (1980) extend the jump diffusion model to allow for several possibly autocorrelated process. Friedman & Laibson (1989) use Press' model to argue in favour of the financial instability hypothesis by Minsky (1977), according to which market participants destabilize the economy by excessive debt-financing of increasingly risky projects in boom economies.

Ball & Torous (1983) suggest a simplified version of Press' model (3.4) to model ordinary and extraordinary price movements, in which they dispend the diffusion term and replace the Poisson mixture of normals with a Bernoulli mixture of normals; that is they allow for only one information event per time interval. The result is a discrete mixture of 2 normals with probability density:

$$f(x) = (1-\lambda)\phi(\mu, \sigma_1^2) + \lambda\phi(\mu, \sigma_2^2) \tag{3.7}$$

where λ denotes the probability of an extraordinary price movement, μ denotes the common drift, and σ_1^2 and σ_2^2 denote the variance of ordinary and extraordinary price shifts, respectively.

This approach has been generalized by Kon (1984) to allow for a discrete mixture of an arbitrary number of normals, where variation in the mean generates skewness, whereas variation in the variance of the components generates excess kurtosis in the resulting probability distribution. Kon motivates the varying parameters in the mixture of distributions by changing regimes in the underlying economy, rather than ordinary and extraordinary information events.

3.2.3 Subordinated Normal Model and Time Changed Brownian Motion

Mandelbrot & Taylor (1967) motivate the use of the Lévy stable distribution for describing the increments in the random walk of logarithmic prices $Z(t) \equiv \ln P_t$ with a non-uniform distribution of trading activity over calendar time t. In order to take this irregularity of transactions into account, they suggest to introduce a randomized operational time $T(t)$ measuring the volume or number of transactions up to physical time t. If $T(t)$ is assumed to follow a Lévy stable distribution with characteristic exponent $\alpha_S < 1$, and increments in $X(v)$, representing price reactions measured in numbers of transactions, are assumed assumed to be iid normal distributed; then it can be shown that the price reaction in calendar time t measured as increments of the process $Z(t) = X(T(t))$ are Lévy stable distributed with $\alpha_S < 1$ despite the normal distribtution of the price reaction conditional on trading volume.

This is a special case of the *Subordinated Normal Model* for logarithmic stock prices, in which transformed calendar time $T(t)$ is subordinated to Brownian motion.[59] A stochastic process $\{X(T(t))\}$ is called *Subordinated* to the process $\{X(t)\}$, if the *Directing Process* $T(t)$ is strincly increasing and has stationary independent increments.[60].

Subordinated Brownian Motion is particularly interesting for modelling stock prices since any arbitrage-free price process may be written as time-changed Brownian motion $B(T(t))$. [61] The chronometer $T(t)$ need however not necessarily be a subordinator, that

[59] see Westerfield (1977).
[60] see e.g. Feller (1966: pp. 333–336).
[61] see e.g. Ané & Geman (2000).

is, it does not need to have stationary independent increments, and $T(t)$ does not need to be independent of the Brownian motion which it is subordinated to[62].

Subordination of Brownian motion implies that the variance of the unconditional return process $B(T(t))$ evolves stochastically, whenever mean or variance of the directing process $T(t)$ are not constant in time[63]. On the other hand, it has been shown e.g. by Barndorff-Nielsen & Shepard (2001) that stochastic volatility models (to be discussed in section 3.3.1) may be written as time-changed Brownian motion with the integrated variance $V_t = \int_0^t \sigma^2(u)\,du$ as independent subordinator. This implies that stochastic variation may be embedded into Brownian motion equivalently by means of a stochastic time change or by stochastic volatility[64].

Clark (1973) shows that whenever the directing process $T(t)$ has finite mean and is subordinated to a process $X(t)$ with finite variance, then the resulting process $X(T(t))$ will also have a finite variance and at the same time exhibit a larger kurtosis than the subordinated process $X(t)$. This provides another avenue to model leptokurtosis in financial returns, without having to resort to infinite variance as suggested by Mandelbrot (1963) and Mandelbrot & Taylor (1967).

There are many possible choices for the directing process $T(t)$.[65] Clark himself suggested T to be lognormally distributed. Praetz (1972) and Blattberg & Gonedes (1974) suggested an inverted gamma distribution and showed that this leads to a scaled *Student t* distribution with probability density function:

$$f(x) = \frac{\Gamma\left(\frac{1+\nu}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\nu}{2}\right)}\nu^{\nu/2}\sqrt{H}[\nu + H(x-m)^2]^{-(\nu+1)/2} \tag{3.8}$$

where $\Gamma(\cdot)$ denotes the gamma function, $m \in (-\infty, \infty)$ is the location parameter, $H \in (0, \infty)$ is the scale parameter and $\nu \in \mathbb{N}$ is the degrees of freedom parameter. The crucial degrees of freedom parameter ν determines the shape of the distribution. The tails of the symmetric Student t distribution decay hyperbolically with exponent ν if it is finite. As ν approaches infinity, the Student t distribution approaches the Gaussian,

[62] see Monroe (1978); Delbaen & Schachermayer (1994); Geman, Madan & Yor (2001).
[63] see Ané & Geman (2000).
[64] see also Barndorff-Nielsen, Nicolato & Shepard (2002) and Barndorff-Nielsen & Shepard (2003).
[65] see Westerfield (1977).

implying exponential decay of the tails.

Hsu (1979b) shows that subordinating Brownian motion to a directing process with exponentially distributed increments, results in a process with increments following the double exponential distribution. This approach may be generalized [66] to yield the *Exponential Power Distributions* (EPD) by Box & Tiao (1973) with probability density

$$f(x) = k_\beta \phi^{-1} \exp\left(-\frac{1}{2} \left| \frac{x-\mu}{\phi} \right|^{2/(1+\beta)} \right) \tag{3.9}$$

where k_β is a normalizing constant, $\phi \in (0, \infty)$ is a scale parameter, $\mu \in \mathbb{R}$ is a location parameter, and $\beta \in (-1, 1]$ is a parameter affecting the shape of the distribution. The EPD are leptokurtic for $0 < \beta \leq 1$, but have tails with either finite endpoints or exponential decline[67].

Madan & Senata (1990) model the variance in driftless Brownian motion to follow a gamma distribution, and call the resulting process *Variance Gamma* (VG). There is no analytical expression available for the probability density of the VG distribution, but it has a very simple characteristic function for the unit period return

$$\varphi_X(u) = \left(1 + \frac{1}{2} \upsilon \sigma^2 u^2\right)^{-1/\upsilon} \tag{3.10}$$

with scale parameter $\sigma^2 \in (0, \infty)$ determining the variance, and shape parameter $\upsilon \in (0, \infty)$ determining the kurtosis of the returns. One attractive feature of the VG model is that, unlike the Student t distribution, it is closed under convolution, thereby allowing returns measured at varying time intervals to be described by members of the same family of distributions. The VG model has later been generalized by Madan, Carr & Chang (1998) in order to allow for skewness in returns. It has finite moments of all orders and exponentially declining tails despite its leptokurtosis.

Unlike Brownian motion, which is a continuous process of unbounded variation, VG is a pure jump process of bounded variation[68]. Carr, Geman, Madan & Yor (2002)

[66] see Hsu (1980, 1982).

[67] see Hsu (1980) and Box & Tiao (1973: pages 156–160).

[68] A function $f : [0, T] \to \mathbb{R}$ is of bounded variation if $\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| < \infty$ for all possible partitions $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$.

generalize the VG process by Madan et al. (1998) further in order to allow for both finite and infinite variation. After ruling out continous processes a priori from the discrete nature of trading, they conclude that equity index returns are better described by jump processes of bounded, than of unbounded variation.

Eberlein & Keller (1995) suggest the *Hyperbolic Distribution* and Barndorff-Nielsen (1997, 1998) the *Normal Inverse Gaussian* (NIG) distribution to model stock returns, both of which are members of the *Generalized Hyperbolic Distribution* family, introduced by Barndorff-Nielsen (1977, 1978) with probability density function

$$f(x) = a_\lambda(\alpha, \beta, \delta)\sqrt{\delta^2 + (x-\mu)^2}^{\lambda-1/2} K_{\lambda-1/2}\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right) e^{\beta(x-\mu)} \tag{3.11}$$

where $K_\nu(\cdot)$ denotes the modified Bessel function of the third kind with index ν; $a_\lambda(\alpha, \beta, \delta)$ is a normalizing constant, α and β are shape parameters in the range $(0 \leq |\beta| < \alpha < \infty)$, $\delta \in (0, \infty)$ is a scale parameter, and $\mu \in \mathbb{R}$ is a location parameter. The parameter $\lambda \in \mathbb{R}$ determines the type of the distribution; the special cases $\lambda = 1$ and $\lambda = -1/2$ correspond to the hyperbolic and normal inverse Gaussian distributions, respectively.

Barndorff-Nielsen (1977, 1978) showed that the generalized hyperbolic distributions may be regarded as variance-mean mixtures of normal distributions, making them candidates for the description of arbitrage-free price processes as well. The normal inverse Gaussian model is particularly appealing, as it is the only subclass of generalized hyperbolic distributions that is closed under convolution. The NIG has, unlike the hyperbolic distribution, a log-density that is concave in the center and convex in the tails, in harmony with empirically observed returns. The tails of the NIG, like those of all generalized hyperbolic distributions, decline however exponentially, just like those of all models described in this subsection except the Student t for finite degrees of freedom parameter ν.

3.2.4 Descriptive Models

While the distributions and processes discussed in the preceeding paragraph have been motivated by economical considerations, such as consistency with arbitrage-free pricing, price response to information flow, or changing regimes in the economy, there have additionally been suggested a large number of data driven models, that came into life solely due to the quality of their fit to empirically observerd return data.

Smith (1981) suggests the *Logistic Distribution* with probability density

$$f(x) = \frac{\exp[(x-\mu)/\alpha]}{\alpha(1+\exp[(x-\mu)/\alpha])^2} \tag{3.12}$$

where $\mu \in \mathbb{R}$ and $\alpha \in (0, \infty)$ are location and scale parameters, respectively. The logistic distribution is symmetric and leptokurtic with exponentially declining tails.

Bookstaber & McDonald (1987) suggest the *Generalized Beta of the second kind* (GB2) as unconditional distribution of financial returns. The 4 parameter family of distributions contains among others the lognormal, the log-Student t, and the log-logistic distributions as special cases. Higher moments may or may not exist, depending upon the values of the shape parameters. The GB2 distributions have been generalized by McDonald & Xu (1995) to the 5 parameter family of *Generalized Beta* (GB) distributions, then containing also the generalized beta of the first kind (GB1), which include among others the Lévy stable distributions as a special case. Exponentiating the GB distributions yields the *Exponential Generalized Beta* (EGB) distributions, which contain among others the exponential power distributions by Box & Tiao (1973).

McDonald & Newey (1988)[69] introduce the 3 parameter family of symmetric *Generalized T* (GT) distributions nesting both the Student t and the exponential power distributions. Theodossiou (1998) generalized GT into the 4 parameter family of *Skewed Generalized T* (SGT) distributions in order to allow for skewness. As said above, neither the EGB nor the SGT family of distributions, like any other distribution discussed in this section, have a foundation in economic theory. Their usefulness is rather due

[69] see also the discussion in Butler, McDonald, Nelson & White (1990).

to the fact that by nesting many different kind of well known distributions, they allow for statistical discrimination between these alternatives for example by means of likelihood-ratio tests.

Badrinath & Chatterjee (1988) suggest to use the so called *$(g \times h)$ Distributions* first introduced by Tukey (1977), for the description of equity returns. One obtains a $(g \times h)$ distributed random variable X by transforming a standard normally distributed random variable Z as

$$X = A + B\frac{\exp(gZ) - 1}{g} \exp\left(\frac{hZ^2}{2}\right) \tag{3.13}$$

where A is a location and B is a scale parameter, and g and h are shape paramaters determining the skewness and kurtosis of the distribution, respectively. The authors apply the $(g \times h)$ distribution to both daily and monthly returns of several US equity indexes[70]. The $(g \times h)$ distribution has also been applied to British stock indices by Mills (1995).

Mantegna & Stanley (1994, 1995) suggest the *Truncated Lévy Flight* (TLF) as a model for arithmetic one minute price changes in the S&P500 index. The truncated Lévy flight is a stochastic process with increments following a rescaled symmetric stable distribution[71] within a finite interval $[-l, l]$, where l denotes the cutoff length, beyond which the density of the increments is set to zero. As such, the TLF looks like a Lévy stable distribution in the center, but has a finite variance due to the cutoff beyond a finite interval. This implies the applicability of the central limit theorem. The authors show, however, that the rate of convergence of the TLF to the Gaussian is about 3 orders of magnitude slower than for most common distributions.

While the VG and generalized hyperbolic distributions are infinitely divisible, TLF is not. This implies by the Khintchine theorem[72] that it may not be thought of as a sum of infinitely many independent, though not necessarily identically distributed, random variables, as would be desirable from an economical point of view[73]. An infinitely

[70] see Badrinath & Chatterjee (1988, 1991).

[71] that is, the location and skewness parameters δ and β in the characteristic function (3.3) are set to zero.

[72] see section 3.2.1 and footnote 52 therein.

[73] see section 3.2.1.

divisible version of the TLF has been given by Koponen (1995), who considers an exponential decay rather than a discontinuous cutoff of the Lévy stable distributed price increments. This *Exponentially Truncated Lévy Flight* is also contained as a special case of the generalization of the VG process by Carr et al. (2002) discussed in section 3.2.3.

3.2.5 Comparison and Evaluation

The comparison of the suggested return distributions in the empirical finance literature yields no coherent picture of superiority for any of the numerous candidates. This may be partly due to the complication arising from the fact that—ecxept for the members of the EGB and SGT families discussed in section 3.2.4—the different distributions are not nested, making statistical inference by means of likelihood ratio tests impossible. Most studies resort then to χ^2 goodness-of-fit tests after arranging the empirically observed return frequencies into class intervals and regarding the values within each interval as a dummy class, and/or use information criteria such as the Schwartz criterion to discriminate between the different candidates.

Praetz (1972) compares the Student t with the Gaussian, Compound Events, and symmetric stable distributions on weekly returns of Australian stocks in 1956–66 and finds the Student t distribution to perform best in χ^2 goodness-of-fit tests. Similarly, Blattberg & Gonedes (1974) find that the Student t distribution fits daily returns better than the symmetric stable distribution for 30 DJI stocks in the period 1957–62; and Kim & Kon (1994) find that the Student t distribution dominates both the discrete mixture of normals and the compound events model in describing daily returns of 30 DJI stocks and 3 stock indexes in the time period 1962–90.

On the contrary, Kon (1984) claimed superiority of the discrete mixture of normals when comparing it to the Student t distribution on similar data, but within the shorter time interval 1962–80, and upon comparing values of their respective likelihood functions. Similarly, Gillemot, Töli, Kertesz & Kaski (2000) compare the Gaussian, the discrete mixture of normals, jump diffusion, Student t, the stable distribution, and

TLF on daily returns of both the Finnish HEX-index and the S&P500. They find the best fit for the discrete mixture of normals, closely followed by jump diffusion and Student t in χ^2 goodness-of-fit tests, whereas the normal and the stable distribution emerge as the worst candidates from their study.

Akgiray & Booth (1987), on the other hand, compare Merton's jump diffusion and Kon's mixture of normals on weekly returns of 200 American stocks and 3 US stock indices and find the former to fit best. Gray & French (1990) compare the normal, Student t, logistic and exponential power distributions on daily returns of the S&P500 and claim superiority of the EPD.

Tucker (1992) compares the likelihood functions of the Student t and general stable distributions, the jump diffusion model, and the discrete mixture of normals on 200 US stocks and 3 stock indices. He finds that the best fit is usually obtained by using either jump diffusion or Kon's mixture of normals. In his study, the Student t distribution is consistently the worst fitting model, due to its inability to model skewness. Peiró (1994), on the other hand, reinforces the case for the Student t distribution, as it obtains the highest scores on the log-likelihood function compared to the general stable, the logistic, the EPD, and the discrete mixture of normals, when applied to 6 international stock market indices.

Barndorff-Nielsen (1997) references a number of studies according to which the normal inverse Gaussian member of the generalized hyperbolic distribution family, discussed in section 3.2.3, fits financial returns better than the hyperbolic distribution suggested by Eberlein & Keller (1995).

Harris & Küçüközmen (2001) fit members of the EGB and SGT distribution families discussed in section 3.2.4 to daily, weekly and monthly returns of both the British FT-SE and the US S&P500 indices in the time period 1979–99. They prefer members of the SGT family for daily and weekly returns, and members of the EGB family for monthly returns of both countries. The fact that the most commonly used distributions are nested within these families[74] allows them to apply likelihood ratio tests in order

[74] see section 3.2.4 above.

to check their adequacy for modelling financial returns. In their study it turns out that many common distributions, such as the Student t, EPD, and the logistic distribution are strongly rejected for daily returns, whereas at least the Student t and the logistic distributions appear acceptable for the description of weekly and monthly returns.

I am not aware of any study that would compare Tukey's $(g \times h)$ distribution to any of the above mentioned distributions, nor any study comparing the best fitting model of the EGB and SGT families with generalized hyperbolic distributions or distributions of the VG type. As said above, there appears no coherent picture from the empirical finance literature regarding the superiority of any one model, except that both the normal and the stable model appear to be inappropriate.

However, it has been pointed out e.g. by Lux & Ausloos (2002), that there is an efficient way to sort out uneligible models by simply considering the behavior of their tails[75]. Discrete mixutes of normals, including the compound events and the jump diffusion model, the VG and the generalized hyperbolic families, Tukey's $(g \times h)$ distribution, as well as TLF have all exponentially declining tails and do thus not qualify as models for stock prices, when a correct description of extremal returns is required.

Ruling out Lévy stable distributions, as they imply infinite variance[76], points to continuous mixtures of normals, such as the Student t distribution and its skewed gerneralizations within the SGT family. This provides, however, only a statistical description, but no economic explanation beyond the mixture interpretation as reaction to incoming news[77]. As such, the success of the statistical approach in identifying an appropriate description of stock returns, even when confined to the unconditional distribution only, appears to be quite limited despite about half a century of intensive research.

[75]see section 2.4.
[76]see section 3.2.1.
[77]see secton 3.2.3.

3.3 Modelling Time-Serial Dependence of Returns

3.3.1 Stochastic Volatility Models

The Geometric Brownian Motion introduced in section 3.1 may be written in differential form equivalently as:

$$dP_t = \mu P_t \, dt + \sigma P_t \, dW_t \tag{3.14}$$

$$d \ln P_t = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma \, dW_t \tag{3.15}$$

where μ and σ denote the instantaneous drift and (constant) volatility, and W_t stands for standard Brownian motion. GBM has become a very popular model of asset returns due to its analytical tractability. The famous option pricing theory by Black & Scholes (1973) for example, assumes stock prices to follow geometric Brownian motion.

Allowing the volatility parameter σ to become a random variable, one obtaines so called *Stochastic Volatility* (SV) models. A discrete time formulation (ignoring drift) is then given by

$$r_t = \sigma_t \cdot \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0,1) \tag{3.16}$$

where r_t denotes the logreturn over one period, and the instantaneous volatility σ_t is a strictly stationary process—often assumed but not necessarily— independent of the iid symmetric noise process ϵ_t. [78]

The first stochastic volatility model has been introduced by Taylor (1986), who assumed $\ln \sigma_t$ to follow an AR(1) process. The earliest continuous time formulation of stochastic volatility is due to Hull & White (1987), who choose the following stochastic processes for the stock price P_t and its instantaneous variance $V_t = \sigma_t^2$:

$$dP_t = \phi(P_t, \sigma_t, t) P_t \, dt + \sigma_t P_t \, dW_t^{(1)} \tag{3.17}$$

$$dV_t = \mu(\sigma_t, t) V_t \, dt + \xi(\sigma_t, t) V_t \, dW_t^{(2)} \tag{3.18}$$

where $W_t^{(1)}$ and $W_t^{(2)}$ denote (possibly correlated) Wiener processes. The fact, that the parameter μ is allowed to depend upon σ_t, allows the inclusion of mean-reverting

[78] see Taylor (1994) and Mikosch (2003b).

volatility into the model. Other SV setups which model volatility clustering more explicitely by introducing mean reversion into the stochastic differential equation for σ_t, include Heston (1993) and Stein & Stein (1991).

Stochastic volatility models in general have the potential to model both skewness, excess kurtosis, and aggregate Gaussianity; since they may just as well be regarded as continuous mixtures of normals with the time dependent integrated variance $V_t = \int_0^t \sigma^2(u)\, du$ serving as the mixing variable[79]. Skewness and kurtosis are then introduced by changing drift and variance, respectively[80], while aggregate Gaussianity is due to ergodicity, as the time averaged integrated variance approaches a constant value for increasing time horizons[81].

Stochastic volatility need not necessarily be embedded in Brownian motion. E.g. Geman et al. (2001) argue that if random time changes (that is stochastic volatility) are related to unforecastable information events, then the time change (or stochastic volatility) should also be purely discountinous, ruling out continous Brownian motion as a model for the resulting price processes as well[82]. This motivated Carr, Geman, Madan & Yor (2003) to suggest several models in which SV is embedded in general Lévy processes[83], as Lévy processes other than Brownian motion are pure jump processes[84].

Long memory has been introduced into stochastic volatiltiy models e.g. by Breidt, Crato & de Lima (1998), who assumes instantaneous volatility to be governed by fractionally integrated Gausssian noise. Barndorff-Nielsen & Shepard (2001) introduce long range dependence by assuming σ_t^2 to be a superposition of Ornstein-Uhlenbeck processes. The unconditional returns follow then a normal inverse Gaussian distribution with exponentially declining tails. Another way to introduce long memory into stochastic volatility is to allow the noise term in (3.1) to be non-normally distributed

[79] see e.g. Barndorff-Nielsen & Shepard (2001) and the discussion in section 3.2.3.

[80] see also section 3.2.2

[81] see Barndorff-Nielsen & Shepard (2003: page 170).

[82] see e.g. Geman et al. (2001: page 82) and Geman (2002: page 1304).

[83] A stochastic process is called a Lévy process if it starts at 0 and has stationary and independent increments.

[84] see e.g. Geman (2002).

and to impose a suitable tail-behavior on ϵ_t[85].

As such, stochastic volatility models exhibit considerable flexibility to model stylized facts of financial returns. Their success has however been hampered by difficulties in estimating the parameters of such models, since volatility is modeled as an unobservable latent process. In particular, SV models in general lack analytical expressions for the one-step-ahead forecasts, which makes estimation by maximum likelihood estimation infeasible[86].

3.3.2 GARCH Models

The specification of *Generalized AutoRegressive Heteroskedasticity* (GARCH) models differs from the stochastic volatility models discussed in section 3.3.1, in as much as the stochastic volatility is fully determined by past realizations of the returns alone. That is, volatility becomes random only through the randomness in the realization of past returns, as there is no extra diffusion term or other source of randomness involved.

Bollerslev (1986) generalized Engle's ARCH(p) specification (2.15) on page 20 into GARCH(p, q) by incorporating the q most recent forecasts for the conditional variance into the current forecast as well:

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j r_{t-j}^2 + \sum_{k=1}^{q} \beta_k \sigma_{t-k}^2 \equiv \alpha_0 + \alpha(L) r_t^2 + \beta(L) \sigma_t^2 \tag{3.19}$$

where again the α_j's and β_k's are non-negative parameters, L is the back-shift operator, $\alpha(L)$ and $\beta(L)$ are the corresponding polynomials in L with coefficients α_j and β_k, and r_t denotes the one-period logreturn defined in (3.16).

Since then there have been many extensions to the GARCH model, most notably the EGARCH model by Nelson (1991) for modelling asymmetric impact of positive and negative shocks, and the ARCH-M model by Engle, Lilien & Robins (1987) to allow for

[85] see Mikosch (2003b).
[86] see e.g. Ghysels, Harvey & Renault (1996) and Shepard (1996).

feedback of the conditional variance into the conditional mean[87]. The most commonly used specification is however GARCH(1,1), as in general it is not outperformed by any of the more sophisticated generalizations[88].

The main advantage of GARCH models is that their parameters—in contrast to those of the stochastic volatility models discussed in section 3.3.1—may be easily estimated by means of conditional maximum likelihood theory, which gives consistent and asymptotically normal parameter estimates, even if the ϵ_t's in (3.16) are not iid normally distributed[89].

Unconditional returns of GARCH processes are leptokurtic and have power-law tails despite their normal building blocks[90], which is at least in qualitatively accordance with empirical evidence. However, residuals from GARCH estimation on financial return series remain usually leptokurtic[91], and maximum likelihood estimation of GARCH parameters imply lighter tails then what is empirically observed[92]. As a solution to the former problem it has been suggested e.g. by Hsieh (1989) and Lye & Martin (1991) to model the distribution of ϵ_t in (3.16) as leptokurtic rather than standard normal. But Pagan (1996) has noted that the potential of this approach may be quite limited, as it may hamper the models very ability to account for dependence in volatility. Another disadvantage of the GARCH model is that the autocorrelation functions of both absolute and squared returns decline exponentially, which implies that GARCH cannot model long range dependence[93].

The GARCH(p, q) model (3.19) has a strictly stationary solution and finite variance if

$$\alpha_0 > 0 \quad \text{and} \quad \sum_{j=1}^{p} \alpha_j + \sum_{k=1}^{q} \beta_k < 1. \tag{3.20}$$

Empirically, however, the sum of the parameters above is usually found to be very close

[87]For surveys on other extensions of the GARCH model, see e.g. the review studies by Bollerslev et al. (1992); Bera & Higgins (1993); Shepard (1996); Gouriéroux (1997).
[88]see e.g. Bollerslev et al. (1992) and Bera & Higgins (1993).
[89]see e.g. Gouriéroux (1997) and Mikosch (2003a).
[90]see de Haan, Resnick, Rootzen & de Vries (1989).
[91]see e.g. Pagan96.
[92]see Stărică & Pictet (1997).
[93]see e.g. Mikosch (2003a) and Mikosch (2003b).

to one in financial data.

Engle & Bollerslev (1986) define a GARCH process as *Integrated in Variance* (IGARCH), if $\alpha(L) + \beta(L) = 1$. Such a specification implies *Persistence in Variance* defined as[94]

$$\limsup_{t\to\infty} |\mathbf{E}(r_t^2|r_0, r_{-1}, \ldots) - \mathbf{E}(r_t^2|r_1, r_0, \ldots)| > 0 \quad \text{a.s.} \tag{3.21}$$

such that shocks to the conditional variance persist indefinitely, which stands in contrast to their exponential decay in the conventional GARCH model. The IGARCH model has a strictly stationary solution, but implies infinite variance, which makes the use of the sample autocorellation function for parameter estimation impossible[95]. Furthermore, as has been discussed already in sections 2.4 and 3.2.1, models implying infinite variance of returns may be safely ruled out based upon tail index values, which for financial returns have been found to be significantly larger than two.

Both conventional GARCH and IGARCH models may be written as ARMA processes in r_t^2:[96]

$$\{1 - \alpha(L) - \beta(L)\}r_t^2 = \alpha_0 + \{1 - \beta(L)\}\nu_t \tag{3.22}$$

with $\nu_t = r_t^2 - \sigma_t^2$ denoting shocks in the conditional variance process. The polynomial $\{1 - \alpha(L) - \beta(L)\}$ has zeros outside the unit circle, unless it is integrated in variance, in which case it contains a unit root. This implies that the IGARCH may equivalently be written as

$$\phi(L)(1 - L)r_t^2 = \alpha_0 + \{1 - \beta(L)\}\nu_t \tag{3.23}$$

with zeros of the polynomial $\phi(L) = \{1 - \alpha(L) - \beta(L)\}(1 - L)^{-1}$ outside the unit circle.

Baillie et al. (1996) introduce the class of *Fractionally Integrated Generalized AutoRegressive Conditionally Heteroskedasticity* (FIGARCH) models as an intermediate model between conventional GARCH and IGARCH by replacing the first difference operator $(1 - L)$ in (3.23) with the fractional differencing operator $(1 - L)^d$ defined as

$$(1 - L)^d \equiv \{1 - dL + d(d-1)\frac{L^2}{2!} - d(d-1)(d-2)\frac{L^3}{3!} + \cdots\}, \quad d \in [0, 1] \tag{3.24}$$

[94]see Bollerslev & Engle (1993).
[95]see Mikosch (2003a).
[96]see e.g. Baillie (1996) and Baillie, Bollerslev & Mikkelsen (1996).

arriving at the following definition of FIGARCH:

$$\phi(L)(1-L)^d r_t^2 = \alpha_0 + \{1-\beta(L)\}\nu_t, \quad \nu_t = r_t^2 - \sigma_t^2 \tag{3.25}$$

with $\alpha_0 > 0$ and zeros of both $\phi(L) = \{1 - \alpha(L) - \beta(L)\}(1-L)^{-1}$ and $1 - \beta(L)$ outside the unit circle. The aim of this model is to replace the exponential decay in the autocorrelation function of conventional GARCH models with hyperbolic decay as empirically oberved in financial markets[97].

The practical relevance of FIGARCH for financial modelling is however still unclear, as the model has been aspersed for not being properly specified[98]. A possible correction has been suggested by Chung (1999). Baillie et al. (1996) claim furthermore that returns following a FIGARCH process have infinite variance and can thus not be covariance stationary. This would however invalidate classical estimation and inference techniques for the same reasons as for the IGARCH model[99].

Similarly, Ding & Granger (1996) aim to introduce long memory into GARCH by modelling the conditional variance as a weighted sum of infinitely many GARCH type variances, which they call *Long Memory (LM)* ARCH. The authors claim a hyperbolic decline of the autocorrelation function, provided that the returns have a finite 4th moment. This statement has however been disproved by Giraitis, Kokoszka & Leipus (2000) for parameter values ensuring stationarity of the model, while for other parameter values it is not yet known whether LM-ARCH has a stationary solution at all[100].

Other recent attempts to include long memory into the GARCH framework include Maheu (2005) and Zumbach (2004). Overall it appears from the discussion above that the GARCH subclass of stochastic volatility models might be less well suited for modeling the stylized facts of financial returns, than what their popularity suggests.

[97] see section 2.6.
[98] see e.g. Chung (1999) and Mikosch (2003a).
[99] see Mikosch (2003a).
[100] see Mikosch (2003a).

3.4 Multifractal Models

Mandelbrot et al. (1997) and Calvet & Fisher (2002) define the *Multifractal Model of Asset Returns* (MMAR) as Brownian motion subordinated to multifractal trading time. That is, they define the log-price process

$$X(t) \equiv \ln P(t) - \ln P(0) = r_t(0) \tag{3.26}$$

on a bounded interval $[0, T]$ and call it MMAR if it adheres to the following assumptions[101]:

1. *$X(t)$ is a compound process*
$$X(t) \equiv B[\theta(t)],$$
where $B(t)$ is a Brownian motion, and $\theta(t)$ is a stochastic trading time.

2. *The trading time $\theta(t)$ is a multifractal process*[102] *with continuous, non-decreasing paths, and stationary increments.*

3. *The processes $\{B(t)\}$ and $\{\theta(t)\}$ are independent.*

Calvet & Fisher (2002) show that if the log-price process $X(t)$ is MMAR, then it is itself multifractal as defined in (2.28). Furthermore it is a martingale, which implies that the discounted price process is arbitrage-free[103].

Multifractality of the trading time $\theta(t)$ is achieved in an iterative process called *Multiplicative Cascade*. The cascade subdivides the interval upon which the multifractal is defined, into smaller and smaller subintervals according to a predefined algorithm, while at the same time distributing probability mass between these subintervals according to another predefined algorithm. The fraction of probability mass in a subinterval

[101]Both studies contain also a more general definition of MMAR as fractional Brownian motion subordinated to multifractal trading time. The limitation to subordinated Brownian motion appears however justified in our context, given the (approximate) martingale property of equity returns (see sections 2.2 and 3.1) and its consistency with arbitrage-free pricing (see section 3.2.3).

[102]see definition (2.28) in section 2.7.

[103]see section 3.2.3.

at iteration k compared to its motherinterval at iteration $k-1$ is called *Multiplier*, resulting in a product of k multipliers for the probability mass in each subinterval, hence the name "Multiplicative Cascade".[104]

The MMAR has finite variance and may or may not have finite higher moments, depending upon the scaling function τ of the multifractal trading time. Despite its finite variance, it cannot have long memory in the sense defined in section 2.6, since it is only defined on a bounded interval. However, its autocovariance in levels, defined as

$$C_q(t) \equiv \mathrm{Cov}(|r_{\Delta t}(t_0+t)|^q, |r_{\Delta t}(t_0)|^q) \tag{3.27}$$

decays hyperbolically in t when $t/\Delta t \to \infty$, a property denoted as *Long Memory in the Size of Increments*. As this is observationally equivalent to long range dependence of $|r_{\Delta t}|^q$ as defined in section 2.6, the MMAR appears to be the first model for an arbitrage-free price process, which is consistent with both fat tails, multiscaling, and long memory. Furhtermore, it is *scale consistent*, that is, in accordance with empirical observations, it describes volatility clustering irrespective of the time scale considered. This property stands in contrast to e.g. GARCH models, which are not closed under temporal aggregation and approach white noise in the limit of infinitely long observation intervals[105].

However, the construction of the MMAR on bounded (though arbitrarily large) intervals implies that price processes defined on an infinite length of time can obey the MMAR only for bounded ranges of time, beyond which they will contain crossovers, that is transitions in their scaling properties. Furthermore, the combinatorial construction of multifractal behaviour is somewhat at odds with the notion of a causal evolvement of prices through time.

This potential drawback is however overcome by its equivalent formulation as a stochastic volatility model[106] with the multipliers interpreted as renewing factors in a so called *Information Cascade*[107]. Müller et al. (1997) show for various exchange rates

[104] For an introduction into the construction of multifractal measures, see e.g. Evertsz & Mandelbrot (1992).
[105] see Diebold (1988), Drost & Nijman (1993), and the discussion in Mandelbrot et al. (1997).
[106] see sections 3.2.3, 3.3.1 and the discussion in Muzy, Delour & Bacry (2000).
[107] The term has been invented by Ghashghaie et al. (1996) as an anology to the Kolmogorov energy

that volatility defined on coarser time scales Granger-causes volatility defined on finer time scales, but not the other way round from fine scaled to longer term volatility. They interpret this as evidence for dissipation of information from long-term to short-term investors through prices. Such a view is confirmed for stock market data by Arnéodo, Muzy & Sornette (1998), who use wavelet analysis to show that coarse-grained volatility predicts fine-scaled volatility of S&P500 index returns, in accordance with the information cascade hypothesis by Ghashghaie et al. (1996) and Müller et al. (1997).

Muzy et al. (2000) provide an example of a multifractal stochastic volatility model based upon the *Multifractal Random Walk* (MRW) introduced by Bacry, Delour & Muzy (2001) and recently generalized by Bacry & Muzy (2003). The MRW generates multifractal behaviour of stock prices within a bounded time interval T by imposing a corresponding correlation structure upon the lognormally distributed stochastic volatilities, while for time scales $\gg T$ the process converges to geometric Brownian motion. Such a model is consistent with the interpretation of T being the information horizon of the longest term investors in the market.

Breymann, Ghashghaie & Talkner (2000) provide the first model, which gives an explicit expression for the k'th *Renewal Probabilities* $a_t^{(k)}$ in the stochastic volatility model (3.16) with[108]

$$\sigma_t = \sigma_0 \prod_{k=1}^{m} a_t^{(k)}, \tag{3.28}$$

thereby allowing for simulations of multifractal processes with explicit reference to the past only, which stands in contrast to the combinatorial construction originally advocated in Mandelbrot et al. (1997).

However, both in the combinatorial and in the stochastic volatility framework, the multipliers or renewal probabilities get updated at fixed points in time, thereby contradicting the notion of randomly arriving information. Calvet & Fisher (2001) randomize this deterministic scheme by assuming an exponential waiting time for the updating of

cascade in thermodynamics, which describes the dissipation of energy injections into turbulent flows from larger to smaller scales.

[108] σ_t denotes the volatility at the shortest time horizon after m renewals of the renewal probabilities $a_t^{(k)}$ at different cascade levels $k = 1, \ldots, m$, and σ_0 denotes the constant volatility at time scales beyond the largest horizon at the top of the cascade.

multipliers in the combinatorial framework, which may then be interpreted as a latent state vector in a Markovian stochastic volatility process. Such Markovian chains allow for volatility forecasting by Bayesian updating. The *Markov-Switching Multifractal* introduced by Calvet & Fisher (2003) allows even for maximum likelihood estimation by interpreting the multipliers in the combinatorial framework as latent volatility state variables in a regime-switching model with identical marginal distribution, but different transition probabilities for each factor.

4 Behavioral Explanations

4.1 Efficient Markets versus Endogeneous Market Dynamics

The efficient market hypothesis (EMH) asserts that the prices of financial assets immediately reflect all publicly available information. It is generally credited to Fama (1965, 1970), but has been mentioned at least as early as 1889 in a book by George Gibson. Formally, it is an application of Muth's earlier mentioned rational expectation hypothesis, which due to the homogeneous expectations of investors results in the so-called rational valuation formula

$$p_t = \sum_{i=1}^{\infty} \delta^i E[d_{t+i}|I_t], \tag{4.1}$$

where p_t denotes the stock price at time t, d_{t+i} is the dividend to be paid in period $t+i$, and $\delta < 1$ is a discount factor. The essentail point is, that the probability measure used in the expectation operator E conditional on the information I_t available at time t, is the same for all investors and coincides with the true probability measure of prices. One may therefore think of (4.1) as linking financial asset prices with the expectations of a single representative agent. The rational valuation formula leads trivially to the martingale property of cum dividend discounted stock prices, which explains much of the popularity of the EMH in the 70th and early 80th.

A central weakness of the EMH is however the lack of any explanation how the price p_t in (4.1) is actually generated by demand and supply, as the immediate incorporation of all value relevant information explicitely excludes any finite price adjustment process. It has even been shown that in a market where all agents are rational and this is commonly known there will be no trade, no matter whether there are information costs involved or not[109]. Trade is however arguably a central feature of any financial market which no model of it should easily dismiss. For example Farmer (2002) notes that trading volume in the foreign exchange markets is at least 50 times larger than world GNP.

[109] see e.g. Rubinstein (1975); Grossman & Stiglitz (1980); Hakansson, Kunkel & Ohleson (1982); Milgrom & Stokey (1982); Geanakoplos (1992).

Lux & Ausloos (2002) note that validity of (4.1) would imply similiar statistical properties of the news arrival process as compared to those of financial returns themselves. This hypothesis appears hardly testable, given the vast ocean of information which might be possibly valuation relevant, but at least there appears to be no evidence of news arriving for example in clusters of high and low volatility.

On the contrary, a number of studies show that the link between publicly available information and financial returns may be weaker than what the EMH suggests. For example, Niederhoffer (1971) investigate 432 significant world event days in the period 1950–66 and find them to be only slightly more likely to show large price movements than other days. Cutler, Poterba & Summers (1989) select 49 major news events in the period 1941–87 and find only a marginal increase in both absolute returns and daily volatility compared to other days. When listing the 50 largest price changes of the S&P 500 stock index they find rarely important news associated with them. Regressions of stock returns upon macroeconomic factors confirm the findings by Fama (1981) and Roll (1988) that it is difficult to account for more than one third of the monthly variation in stock returns on the basis of systematic economic influences.

The key assumption leading to market efficiency is that intense competition between market participants will eliminate irrational speculators and has been mentioned alredy by Kaldor (1939). The logic is neatly summarized in Cootner:

> If any group of investors was consistently better than average in forecasting stock prices, they would accumulate wealth and give their forecasts greater and greater weight. In the process they would bring the present price closer to the true value. Conversely, investors who were worse than average in forecasting ability would carry less and less weight. If this process worked well enough, the present price would reflect the best information about the future in the sense that the present price, plus normal profits, would be the best estimate of the future price. (Cootner 1964: page 80.)

The theoretical argument for this is given by Friedman, who claims that destabilizing speculation (trades which move prices away from fundamental value) is on average accompanied by a loss:

> I am very dubious that in fact speculation in foreign exchange markets would be destabilizing... People who argue that speculation is generally destabilizing seldom realize that this is largely equivalent to saying that speculators lose money, since speculation can be destabilizing in general only if speculators on the average sell when the currency is low in price and buy when it is high. (Friedman 1953: page 175.)

The popularity of the EMH over several decades documents the strong intuitive appeal of Friedmans hypothesis, as it was widely accepted even though the first counterexample of profitable destabilizing speculation was presented just a few years later by Baumol (1957). Just when the EMH became the main paradigm due to the work of Fama (1965, 1970), Schimmler (1973) showed building on the work of Farrel (1966) that Friedmans hypothesis holds only for the special case when non-speculative demand is a linear function of current mispricing alone. That is, speculation may in general very well drive prices away from fundamental value and yet be profitable. Informationally efficient markets would therefore require price stabilizing arbitrage to be generally more profitable than destabilizing speculation.

A number of studies point however at the limits of arbitrage due to their riskiness and capital constraints of the arbitrageur. For example DeLong, Shleifer, Summers & Waldman (1990) present a model with a risky and a riskless asset in which uncertainty about future opinions of noise traders limits readiness of rational traders to arbitrage and creates "noise trader risk" which drives the price of the risky asset down, and thus its expected return up. If noise traders are bullish on average, they are ready to hold more of the risky asset and earn thus higher returns than the rational arbitrageur. Risk aversion may therefore prevent rational traders from taking over the price dynamics in this model. Further studies which demonstrate and discuss the limits of arbitrage include Russel & Thaler (1985); Black (1986); LeRoy (1989); Shleifer & Summers (1990); Shleifer & Vishny (1997) and Thaler (1999). This opens up the possibility of financial markets dominated by short term rather than long term investors, a point which has been made much earlier by Keynes:

> It might have been supposed that competition between expert professionals, possessing judgement and knowledge beyond that of the average private investor, would correct the vagaries of the ignorant individual left to himself. It happens, however, that the energies and skill of the professional investor and speculator are mainly occupied otherwise. For most of these persons are, in fact, largely concerned, not with making superior long-term forecasts for the probable yield of an investment over its whole life, but with foreseeing changes in the conventional basis of valuation a short time ahead of the general public. (Keynes 1936: page 154.)

Even when arbitrageurs dominate the market, they are more likely to adapt a set of simple strategies than to agree on true values of investments or true probability distributions of stock prices. This point was made already by Alchian (1950) who stresses the importance of positive ex-post profits in contrast to rationally maximized ex-ante profits in the evolutionary struggle for survival. Contrary to the EMH, full knowledge of the economy is not necessary to survive, as

> positive profits accrue to those who are better than their actual competitors, even if the participants are ignorant, intelligent, skilful, etc. The crucial element is one's aggregate position relative to actual competitors, not some hypothetically perfect competitors. (Alchian 1950: page 213.)

Uncertainty created by a highly complex environment favors then adaptive, simple rules of thumb rather than rationally maximizing behaviour (Alchian 1950: p. 218). The latter point was extensively elaborated by Simon (1957), who showed that decision makers in a variety of context act what he termed “boundedly rational” in the sense that they systematically restrict the use and acquisition of information compared to that potentially available.

Their view was supported by evidence from psychology laboratory experiments such as Kahneman & Tversky (1973) and Tversky & Kahneman (1974, 1981) showing that human beings do not behave rational under uncertainty but use simple heuristics which may lead to significant biases even in simple decision problems[110]. A possible reason for this has been worked out by Heiner (1983), who establishes formally in a general evolutionary context, that awareness to a specific kind of information will not be beneficial for survival unless its reliability exceeds a given threshold. Increasing complexity of the environment diminishes in general the reliability of most specific pieces of information and reduces thus the flexibility of the evolutionary surviving agents, as they take less information into account than they potentially could. All this points towards an ecology of agents with simple heuristic strategies rather than a single representative agent with unlimited capacity to immediately assess all relevant publicly available information in a correct way.

Proponents of the EMH such as Lucas (1986) and Rubinstein (2001) have frequently defended their position by claiming that while all investors need not be rational, prices were still set as if all investors had rational expectations:

> Each investor, using the market to serve his or her own self-interest, unwittingly makes prices reflect that investor's information and analysis. It is as if the market were a huge, relatively low-cost continuous polling mechanism that records the updated votes of millions of investors in continuously changing prices. In light of this mechanism, for a single investor (in the absence of inside information) to believe that prices are significantly in error is almost always folly. (Rubinstein 2001: page 19.)

Despite its intuitive appeal, for a proper evaluation of such arguments it appears necessary to take the diversity of traders expectations and investment strategies explicitly into account. Kirman (1992) writes about this topic:

[110]see also the discussions in Simon (1979); Arrow (1982); Kahneman (2003). For evidence of non rational behaviour in experimental asset markets see e.g. Smith, Suchanek & Williams (1988) and Sunder (1995).

> The "representative" is used to *provide* the stability and uniqueness of equilibria which are not *guaranteed* by the underlying model. This applies to the standard suggestions that well-informed individuals are constantly doing the necessary arbitrage to bring the economy back to its equilibrium. If this is the case, individuals must differ, at least in their information. Once this is so, one has again to *prove* that the arbitrage activity will lead back to equilibrium. As Stiglitz (1989) points out, simply to assume this is wholly unwarranted. (Kirman 1992: page 120.)

As pointed out by Arthur (1995), allowing for heterogenous expectations necessarily implies that traders form their expectations by inductive reasoning, since rational deduction of their expectations would lead them into an infinite loop of forming unbiased predictions of all other agents expectations in the spirit of Keynes "beauty contest". While models of learning may in certain situations converge to rational expectations equilibriua[111], they require a reward for correct forecasts and structural stability of the learning environment. Given the discussion above, whether those assumptions apply to real financial markets for assets with intrinsic values depending upon dividends in the unknown future of a constantly changing environment, remains an open issue.

In reality we know that people engage into all sorts of trading strategies and that the diversity of opinions regarding investment opportunities is in general large rather than small.[112] It appears then that the representative agent limits rather than enhances our understanding of financial markets, as it by definition eliminates all interaction, in particular trade, between market participants. The shortcomings of the representative agent approach are extensively discussed e.g. in Kirman (1992) and Ramsey (1996), who both stress that in general one may neither expect a functional nor a parametric relationship between aggregate behavior and that of the individual agents making up the aggregate, simply because the description of individuals does not take their interaction into account. Ramsey (1996) argues therefore for a mass-statistical description of the economy, where macrovariables at the aggregate level are defined in terms of

[111] see e.g. Evans & Honkapohja (2001).

[112] refer for example to the large trading volume in financial markets or survey studies among financial specialists such as Allen & Taylor (1990); Frankel & Froot (1987a,b, 1990a,b); Taylor & Allen (1992); Lui & Mole (1998); Menkhoff (1998) and Cheung, Chinn & Marsh (2000).

self-contained differential equations, which define “laws of motion” for the economy as a whole. These are obtained as approximations from a probabilistic description of the economy in terms of the so called *master equation*

$$\frac{\partial P(y,t)}{\partial t} = \int \left[W(y|y')P(y',t) - W(y'|y)P(y,t)\right] \, dy', \tag{4.2}$$

where y_t is a Markov process describing the evolution of the relevant macro variable (e.g. the price of a financial asset) in time, $P(y,t)$ is the corresponding time varying probability function for some sub-process initiated at time t_0 at initial condition y_0, and $W(y'|y)$ is the per unit time transition probability function from state y to state y'. The Master equation is a “gain-loss” equation in which the change in the probability distribution for y_t depends on the transitions of states into y less the transitions out of state y, each being weighted by the current probability of being in the relevant state. Interactions at the micro level between individual agents have to be subsumed into the transition rates W which, in contrast to the representative agent approach, provide the core of the dynamics in the macro variabel y_t.

The master equation approach has originally been developed as a device in statistical physics to derive the properties of physico-chemical multi-coponent systems on the macroscopic level from their constituent components on the elementary microscopic level. In general, the interaction between microscopic units leads to “emergent” properties of the macroscopic system in the sense that the properties of the aggregate system are *fundamentally different* from those of its constituents. For example, the classical mechanics law of motion of a particle are all time reversible, whereas the macro relationships derived from them, such as the heat exchange between a hot and a cold body, are not. Similar situations in which a system composed of many parts or individuals acquires a new structure on a macroscopic scale, occur also in other fields of the natural sciences such as chemistry and biology. This led to the introduction of a new interdisciplinary branch of science called “synergetics”, defined as the science of collective phenomena in systems with “cooperative” interactions occuring between the units of the system[113].

The field of synergetics has been extended to the social sciences by Weidlich & Haag

[113]see e.g. Haken (1983).

(1983)[114] motivated from the observation that due to the approximation schemes involved in the master equation approach and the related mean field theory, a probabilistic description of the motion of macrovariables proved to be applicable even when the details of the microfluctuations of the system are unknown.[115] Just like physico-chemical systems are composed of a large number of particles, each of them existing in one of several possible states, a society may be regarded as being composed of a large number of members, who individually adopt different attitudes or "states" of behaviour. Therefore a probabilistic description of decision processes in a society might also prove to be adequate, where the change in attitude of its members are subsumed in corresponding transition probabilities of the macro variables.

This provides an avenue to approach the statistical properties of financial time series as the result of an endogeneous dynamics originating from the interaction of individual traders (denoted as *Interacting Agent Hypothesis* in Lux & Marchesi (1999)) rather than to postulate their existence already in the unobservable news arrival process as in the representative agent approach of the efficient market framework. Support for this view is given by the fact that statistical properties of stock returns such as absence of serial correlations, heavy tails with power law decay, volatility clustering, long memory, multiscaling, and a positive corellation between trading volume and return variance are common to returns of every actively traded financial asset. This is why they have become known as so called *Stylized Facts*, which every viable statistical model of asset returns should be able to generate[116]. As noted in Lux & Ausloos (2002), the interacting agent hypothesis would allow to explain this perplexing similarity of the statistical characteristics of very different markets by the similiarity of the behaviour of traders, reminiscent of the occurrence of universal scaling laws in many-particle systems independent of the microscopic details of the system.[117]

[114]see also Weidlich (1991, 2002).
[115]see Ma (1976) for an introduction to mean field theory and its use in explaining universal scaling.
[116]see e.g. Pagan (1996); Cont (2001) and Lux & Ausloos (2002). The term itself is due to Kaldor (1961).
[117]see e.g. Ma (1976) for an introduction to universal scaling laws in thermodynamics, and Stanley et al. (1996) for a discussion of their possible connections with economics.

4.2 Dynamic Models of Fundamentalist Chartist Interaction and Mimetic Contagion

Survey data on exchange rate expectations of professional traders shows consistently that financial specialists regard charts of past prices as an important source of information beyond the analalysis of economic fundamentals, in particular at short horizons [118]. This is even true when traders regard themselves as fundamental investors, as is evident from the following quotation of a respondant in Taylor & Allen (1992):

> Knowledge of chart signals is essential to all operators as they have a bearing on the action of many market participants ... This holds true both for operators who place high priority on technical analysis and for others—like ourselves—who prefer a more fundamental approach.
> (Taylor & Allen 1992: page 311.)

The latter statement highlights that technical analysis or chartism, that is the search for patterns in the time series of historical prices in order to generate a price forecast, is intimately connected with herding, that is the imitation of other investors trades regardless of ones own beliefs and information. Reasons for mimetic contagion among asset managers include, among others, the desire to infer information from other investors trades (information based herding), the desire of managers to show quality in particular in the context of short mandates and frequent performance checks—often on peer group benchmarkts or capitalization weighted indices (reputation-based and compensation-based herding), and dynamic hedging in so-called contingent immunization or portfolio insurance strategies, which manage portfolios containing risky assets based upon their recent performance[119].

[118] see the survey studies cited in footnote 112 on page 66. Evidence for profitability of technical trading is provided e.g. in Brock, Lakonishok & LeBaron (1992); Jegadeesh & Titman (1993); Chan et al. (1996); Caginalp & Laurent (1998) and Hogan et al. (2004). Critical discussions regarding out-of-sample performance can be found in Sullivan et al. (1998, 1999) and Schwert (2003).

[119] see e.g. Bikhchandani & Sharma (2001); Davis (2003) and Hirshleifer & Teoh (2003) for literature overviews, and Caparrelli, D'Arcangelis & Cassuto (2004); Hwang & Salmon (2004); Sias (2004); Walter & Weber (2006) for recent evidence.

In the following we shall take a brief look at some microscopic mdels of heterogenous interacting agents, which bear some relation to the model to be developed in section 5, in that they aim to explain the statistical properties of financial time series from some kind of fundamentalist chartist interation or mimetic contagion, and that at least some qualitative results can be deduced by means of analytical methods. This is only a small subset of the literature on the dynamically evolving field of heterogenoeus agent models, which has already grown too large to be comprehensively reviewed here. For more extensive reviews on such models refer to Tesfatsion & Judd (2006), in particular chapters 8 and 9.

4.2.1 Fundamentalist Chartist Interaction

The first quantitative model of fundamentalist chartist interaction is due to Zeeman (1974) in an effort to model bubbles and crashes in a stock market. Zeeman describes the stock market in terms of a stock index and the excess demand for stocks by fundamentalist and chartists. He shows that purely qualitative assumptions about the interplay of these three variables suffice to explain cycles of bull and bear markets or market crashes depending upon the proportion of the market held by chartists. The basic mechanism for generating such dynamic behaviour of the stock market is the same as in nowadays' models: A stochastic disturbance of the equilibrium price generates self-accelerating excess demand by chartists, until the price is sufficiently far away from equilibrium to be corrected by fundamentalists.

Beja & Goldman (1980) provide the first explicit formalization of trading demand by chartist and fundamentalists and show that a large excess demand by chartists relative to fundamentalists may destabilize an otherwise stable price equilibrium. The excess demand D_t^f by fundamentalists is formalized as

$$D_t^f = a(p_f(t) - p(t)), \qquad a > 0, \tag{4.3}$$

where $p_f(t)$ and $p(t)$ denote the exogenously generated fundamental price and the endogeneously determined trading price, respectively, and the coefficient a measures the relative impact of fundamental demand upon price movements.

Chartists use a technical indicator of the prevailing price trend $\psi(t)$, which they compare to the exogenously given return on an alternative security $g(t)$. Their excess demand D_t^c is assumed to depend linearly on the return differential between both securities, that is,

$$D_t^c = b(\psi(t) - g(t)), \qquad b > 0, \tag{4.4}$$

where the coefficient b measures the relative impact of chartist demand upon price movements. The trend estimator ψ is adaptively adjusted to the real price trend according to the differential equation

$$\dot{\psi}(t) = c[\dot{p}(t) - \psi(t)], \qquad c > 0, \tag{4.5}$$

where the dots denote derivates with respect to time and c is the adaption speed.

In an equilibrium setting with equal demand and supply of shares the trading demands by chartists and fundamentalists would have to add up to zero. But Beja & Goldman allow explicitly for disequilibrium trading by assuming a finite adjustment speed of the trading price in the direction of net asset demand

$$\dot{p}(t) \propto D_t^f + D_t^c, \tag{4.6}$$

such that

$$\dot{p}(t) = a(p_f(t) - p(t)) + b(\psi(t) - g(t)) + e(t), \tag{4.7}$$

where the speed of price adjustment has been absorbed into the parameters a and b and $e(t)$ denotes an additional noise term. It is then shown that the system of differential equations consisting of (4.5) and (4.7) is stable with p converging to p_f if and only if

$$a > c(b - 1) \tag{4.8}$$

and becomes unstable with exploding price oscillations otherwise. That is, a large impact of fundamental demand a acts in a stabilizing manner, whereas both a large impact of chartist demand b and a high price trend adaption speed c of the speculators tends to destabilize the market. Chiarella (1992) povides a nonlinear generalization of the chartist excess demand D_t^c, for which the exploding price oscillations in Beja & Goldman in the unstable case are replaced by a stable limit cycle along which prices fluctuate without ever converging to fundamental value.

Sethi (1996) extends that model further by introducing information costs for fundamentalists and explicitely considering the fluctuations in wealth and inventories of the two trader types. While chartists would loose their money to fundamentalists under absence of information costs, this is no longer the case when information about fundamentals is costly to obtain. The relative profitability of the two trading strategies depends then upon whether the market is within the stable or the oscillatory regime. In the oscillatory regime with large deviations between market and fundamental price the fundamentalist approach remains the more profitable investment style, whereas in the stable regime with prices near fundamental value profits are not sufficient to cover their information costs and chartism becomes more profitable. Chartists and fundamentalists entertain a symbiotic relationship in the sense that increasing wealth of chartists pushes the market into the oscillatory regime which fundamentalsits need in order to make their profits by driving the market back into the stable regime. As a result, the market alternates continously between periods of stability and instability.

This is similiar to the exchange rate model by DeGrauwe, Dewachter & Embrechts (1993), in which endogenously changing weigths of chartists and fundamentalists may generate periodic and even chaotic fluctuations of the exchange rate. Like in Sethi, the fraction of chartists increases endogenously with the mispricing of the foreign currency. This is however not motivated by wealth shares as in Sethi, but with offsetting trades of fundamentalists near the fundamental equilibrium exchange rate.

Chaotic price fluctuations are also generated in a non-linear variant of the Beja & Goldman model in discrete time by Day & Huang (1990). They provide also a justification of the price adjustment rule (4.6) in terms of a market maker. The market maker supplies stocks out of his inventory and raises the price if there is excess demand, while he accumulates stock to his inventory and lowers the price when there is excess supply. Even though Day & Huang do not explicitely model the market makers inventory, they do stress the importance of keeping the latter in balance in order to ensure successful operation of the market pricing mechanism.

Farmer (2002) and Farmer & Joshi (2002) provide the following derivation of an approximately linear relationship between asset returns and net asset demand D_t from all traders, which they call the *market impact function.* Farmer & Joshi assume that the relative increase of the price p from period t to period $t+1$ is an increasing function ϕ of current demand alone:

$$p_{t+1}/p_t = \phi(D_t) \quad \text{with} \quad \phi' > 0, \quad \phi(0) = 1. \tag{4.9}$$

Taking logarithms and expanding in a Taylor series yields

$$\ln(p_{t+1}) - \ln(p_t) \approx D_t/\lambda, \tag{4.10}$$

where $\lambda := 1/\phi'(0)$ measures the market depth or liquidity. Note that (4.10) is the discrete time analogue to the price adjustment mechanism (4.6) by Beja & Goldman, except that it holds for logarithmic rather than raw price changes. However, Farmer & Joshi point also at the following weakness of demand functions such as (4.3) in Beja & Goldman. If traders continously issue new trades as long as a mispricing exists, they are at risk of building up unbounded inventories, because in general their position is not forced to go to zero, even when mispricing goes to zero. The notion of traders having a non-zero or even unbounded exposure to market risk when they believe the market is fair priced, is however both counterintuitive and unacceptable from a risk management point of view.

Brock & Hommes (1998) introduce *Adaptive Belief Systems* as a mechanism of endogenous predictor choice in a market with heterogeneous expectations, which does not depend on disequilibrium trading with prices set by a market maker. Instead they assume that each investor type h is a myopic mean variance maximizer, such that her demand for shares z_{ht} is given by

$$z_{ht} = \frac{E_{ht}(p_{t+1} + d_{t+1} - (1+r)p_t)}{\gamma V_{ht}(p_{t+1} + d_{t+1} - (1+r)p_t)}, \tag{4.11}$$

where p_{t+1} and d_{t+1} denote the stochastic price and dividend of the next period, r and p_t are the risk free rate and the current price, γ is a risk aversion parameter, and E_{ht} and V_{ht} denote the subjective beliefs of investor type h about the conditional expectation and conditional variance using her indiviudal information set at time t. Note that z_{ht}, in contrast to the demand functions D_t in Beja & Goldman and Farmer & Joshi above, denotes target holdings rather than trading demand, as it is derived from maximization of absolute utility. This will coincide with trading demand only if the trader has no prior position.

Summation over all trader types yields for the aggregate market demand z_t

$$z_t = \sum_{h=1}^{H} n_{ht} \frac{E_{ht}(p_{t+1} + d_{t+1} - (1+r)p_t)}{\gamma V_{ht}(p_{t+1} + d_{t+1} - (1+r)p_t)}, \tag{4.12}$$

where n_{ht} denotes the fraction of agents using predictor h and H is the number of trader types. Each predictor gets assigned a fitness function U_{ht}, which is a performance measure of past realized profits from using predictor h. Brock & Hommes introduce an endogenous selection mechanism of forecasting rules by updating the fractions of type h agents n_{ht} using the *multinomial logit model* of discrete choice[120]

$$n_{ht} = \exp(\beta U_{ht})/Z_t, \tag{4.13}$$

where β is the *intensity of choice* measuring how fast agents switch between different prediction strategies, and Z_t is a normalization factor. Brock & Hommes provide several examples in which rational traders fail to drive noise traders out of the market, even when there are no information costs for fundamentalists. Chiarella, Dieci & He (2001) and Gaunersdorfer (2001) present adaptive belief systems in which switching between multiple price equilibria leads to leptokurtic price series with volatility clustering. The latter is extended in Gaunersdorfer & Hommes (2007) to produce also long memory in volatility.

Chiarella, Dieci & He (2003) allow for disequilibrium trading by introducing a market maker, who adjusts the trading price proportional to the aggregate market demand in (4.12) according to

$$p_{t+1} = p_t + \mu z_t, \tag{4.14}$$

where μ denotes the speed of price adjustment. Chiarella et al. do not model the market makers' inventories, but given that their price adjustment is proportional to target holdings rather than trading demand as generated from the adjustment of such holdings, it is to be expected that the criticism by Farmer & Joshi (2002) applies to their model as well. In a later paper (Chiarella, Dieci & He 2006), they extend their model to the case of two risky assets. The focus in both papers is on establishing some necessary and/or sufficient conditions for the stability of the "fundamental" equilibrium

[120] see Manski & McFadden (1981) for an extensive overview of discrete choice models and Brock & Hommes (1997) for a motivation of discrete choice models as endogemous coupling mechanisms between market equilibrium dynamics and predictor selection.

rather than generating time series properties in accordance with the stylized facts of financial returns.

Westerhoff (2004) presents the only study of a multiasset market I am aware of, which produces return series in accordance with the main stylized facts, uncorrelated returns with volatility clustering, fat tails of the return distribution with power law scaling, and long memory in volatility. He assigns a fitness function to each traded asset, that measures the attractiveness for chartists to trade in that asset, in a similar spirit as the chartist weight was determined in the exchange rate model by DeGrauwe et al. (1993). The further the asset price deviates from its intrinsic value, the less attractive trading in that asset becomes to the chartist, as the risk of being caught in a bursting bubble increases. Westerhoff uses a discrete-choice model of the form (4.13) to determine endogenously the weights of chartists in trading the different assets in the market. He does not discuss correlation between asset returns, but shows that dispersion between prices of assets with identical intrinsic value decreases, when traders condition on the same information. Westerhoff applies the market maker approach using the loglinear price impact function (4.10) of Farmer & Joshi (2002) in an order-based setup of the same spirit as Beja & Goldman (1980), without modelling market makers' inventories. That is, traders issue continuously new orders as long as their target price is not reached.

4.2.2 Mimetic Contagion

The first formal model of herding in economics which I am aware of has been presented by Föllmer (1974), who uses the Ising model from statistical physics in order to describe the choice of agents between two commodities. The Ising model explains so called *critical phenomena* or *phase transitions*—sudden changes of macroscopic systems as the result of arbitrary small changes in a thermodynamic variable—in terms of the energy difference between identical and opposing states of neighbouring microscopic units placed upon a lattice. Examples of critical phenomena include spontaneous magnetization and transitions between the solid, liquid, and gaseous phases or their coexistence as a function of temperature. Föllmer identifies the two possible states of the microscopic units with a preference of economic agents for either the first or the second good in an exchange economy, and the energy difference with their propensity to go with or against the trend as reflected in the states of their neighbours. Critical phenomena reappear then as the breakdown of price equilibria for strong and complex enough interaction between the economic agents. Furthermore, information about the microscopic states may not be suffiecient to determine the macroeconomic phase, that is the probability laws of individual agents may in general differ from the global probability law governing the joint behaviour of all economic agents.[121]

Similiar lattice based models have been employed e.g. by Cont & Bouchaud (2000) and Iori (2002) in order to explain some of the statistical properties of financial returns. Both consider three-state models, in which agents may either buy, sell or choose not to trade. Denoting the demand of agent i out of N agents with $\phi_i \in \{-1, 0, +1\}$ for sell/inactive/buy, applicaton of the log-linear price impact function (4.10) yields for the logreturn $r(t)$

$$r(t) = \frac{1}{\lambda} \sum_{i=1}^{N} \phi_i(t), \tag{4.15}$$

where λ is a measure of liquidity, as before. Cont & Bouchaud (2000) consider the bond percolation model[122], in which bonds between nearest neighbours are occupied with probability p. This model is characterized by a percolation threshold p_c, such that for $p < p_c$ the system decomposes into disconnected clusters of the same state, whereas for $p > p_c$ an infinite cluster occurs. It is known that the number of clusters n_s containing s units decreases near the percolation threshold $p = p_c$ as a power-law:

$$n_s \propto s^{-\tau}, \tag{4.16}$$

with an exponent τ between 2 and 2.5 depending upon the dimensions of the lattice. Interpreting the units as traders and keeping in mind that the sum of identically behaving agents in (4.15) is just the sum of the corresponding clusters times their respective size s, yields for the market return a power law with exponent $-(\tau - 1)$. While such an exponent appears somewhat too low to be consistent with real financial market returns, Sornette, Stauffer & Takayasu (2002) discuss several extensions of the basic model, which bring its value closer to the commonly observed tail index $\alpha \approx 3$.

[121] see the discussion of the representative agent approach in section 4.1.

[122] see e.g. Sahimi (1994)

Iori (2002) comes even closer to a replication of the stylized facts of financial returns by considering a superposition of a lattice based communication structure and idiosyncratic signals, where the sum of both has to exceed an agent-specific threshold in order to generate a trading order, with thresholds periodically adjusted proportional to price changes. Iori's model simultaneously generates uncorrelated returns with volatility clustering, long memory in volatility, a positive cross-correlation between volatility and trading volume, and power-law distributed returns with a realistic tail index.

Kirman (1991) provides an exchange rate model, which combines an infection process inspired from the communication behaviour of tandem recruiting ants (Kirman 1993) with chartist-fundamentalist interaction of utility maximizing agents. Traders hold either a chartist or a fundamentalist view of the exchange rate and meet at random in discrete time. When two agents meet, the first is converted to the seconds view with a given probability $(1 - \delta)$. There is also a small probability ϵ of spontaneous change in opinion in order to avoid absorbing states with all agents holding the same view. Such an infection process may be described as an ergodic Markov chain with a symmetric bimodal limit distribution for small enough spontanoues conversion probability ϵ compared to the infection probability (1-δ), with maxima near the extremes of identical opinion of all agents. That is, the investment community spends most of the time holding either a chartist or a fundamentalist view of the exchange rate, with only occasional shifts between both regimes. With chartists extrapolating the recent price trend, fundamentalists expecting reversion to the fundamental price, and a price equilibrium equiation derived from mean-variance utility maximization of both agents, prices are close to fundamental value when fundamentalists dominate, but follow bubble paths under the chartist regime. The endogenous switching between both regimes induced by the infection process above implies then a near unit-root process with clustered volatility. In a later extension (Kirman & Teyssière 2002), the model generates also long memory in volatility.

It is an important advantage of formulating agent-based models as ergodic Markov chains with explicit limit distributions, that it allows for estimation of the underlying parameters by comparison of the model implied return distribution with the returns observed in financial markets. This has been done for Kirman's original model by Gilli

& Winker (2003) and for an extended version with asymmetric transition probabilities by Alfarano, Lux & Wagner (2005). Another advantage is that it may help to discern true scaling, that is, the emergence of power laws over all orders of magnitude, from spurious multiscaling only in the pre-asymptotic regime. Alfarano & Lux (2006) consider a simplified variant of the noise trader infection model by Lux & Marchesi (2000) to be discussed in the next section, simple enough to be formulated as an ergodic Markov chain with an analytical accessible limit distribution. While the true process is a stationary stochastic volatility process with finite third and fourth moments and exponentially declining autocorrelation in volatility, the authors are able to demonstrate apparent power law scaling in volatility and tail indices near 3 in the pre-asymptotic regime of a few thousand observations, a common sample size in empirical investigations of financial returns. The authors attribute this apparent scaling reminiscent of a stochastic volatility model with apparent multiscaling by LeBaron (2001) and a short memory model with apparent long memory by Granger & Teräsvirta (1999), to switches between high and low volatility regimes as suggested e.g. by Stărică & Mikosch (2000); Diebold & Inoue (2001) and Mikosch & Stărică (2004), here due to temporary dominance of either chartists or fundamentalists in the market.

Consider finally the herding model by Lux (1995) as an introduction to the next section. He considers an investment community of $n_+ + n_- = 2N$ speculators with n_+ optimists (buyers) and n_- pessimists (sellers). The configuration of the investment community is then uniquely specified in terms of the state variable

$$n := \frac{1}{2}(n_+ - n_-) \qquad \text{with} \qquad -N \leq n \leq N. \tag{4.17}$$

Lux models the population dynamics as a Markov process, in which $P(n;t)$ denotes the probability of finding the investment community in state n at time t, applying the master equation approach. Because n is a discrete variable, the master equation (4.2) reduces to

$$\frac{d\,P(n;t)}{dt} = \sum_{n'} \left[w(n|n')P(n';t) - w(n'|n)P(n;t) \right], \tag{4.18}$$

where $w(n'|n)$ denotes the per unit time transition probability from state n to n'.

Changes in the configuration of the investment community are governed by switches of individual agents between the optimist and pessimist subgroups. Denote the state dependent probability of a single pessimist to become an optimist per unit time with $p_{-+}(n)$, such that the event of a pessimist becoming an optimist within the time interval Δt is Bernoulli distributed with probability $p_{-+}(n)\Delta t$. The total number of pessimists changing to an optimistic view of the market within that time interval is then binomially distributed with parameters n_- and $p_{-+}(n)\Delta t$, which in the limit of a large population and a small time interval becomes Poisson distributed with parameter $n_- p_{-+}(n)\Delta t$. The probability of an integer increase $\Delta n = 1, 2, \ldots$ of the state variable n over the time interval Δt is therefore given by

$$P(n + \Delta n; t + \Delta t | n; t) = \frac{(n_- p_{-+}(n)\Delta t)^{\Delta n}}{\Delta n!} e^{(n_- p_{-+}(n)\Delta t)}, \tag{4.19}$$

such that defining the per unit time transition probability from state n to state $n + \Delta n$ as

$$w(n + \Delta n | n) := \lim_{\Delta t \to 0} \frac{P(n + \Delta n; t + \Delta t | n; t)}{\Delta t}, \tag{4.20}$$

one obtains for the transition rate from state n to $n + \Delta n$:

$$w(n + \Delta n | n) = n_- p_{-+}(n) \delta_{\Delta n, 1}, \qquad \Delta n = 1, 2, \ldots, \tag{4.21}$$

where δ denotes the Kronecker delta function

$$\delta_{x,x'} := \begin{cases} 1, & \text{if } x' = x; \\ 0, & \text{otherwise.} \end{cases} \tag{4.22}$$

In a similar way it can be shown that the transition rate from state n to state $n - \Delta n$ is given by

$$w(n - \Delta n | n) = n_+ p_{+-}(n) \delta_{\Delta n, 1}, \qquad \Delta n = 1, 2, \ldots, \tag{4.23}$$

where $p_{+-}(n)$ denotes the state dependent transition probability per unit time of an individual agent to move from the optimist to the pessimist subgroup. The master equation contains therefore only transitions between nearest neighbour states n and $n' = n \pm 1$. Abbreviating

$$w_{-+}(n) := n_- p_{-+}(n), \qquad w_{+-}(n) := n_+ p_{+1}(n), \tag{4.24}$$

the master equation (4.18) reduces to

$$\begin{aligned} \frac{d\,P(n;t)}{dt} &= w_{-+}(n-1)P(n-1;t) + w_{+-}(n+1)P(n+1;t) \\ &\quad - w_{-+}(n)P(n;t) - w_{+-}(n)P(n;t). \end{aligned} \tag{4.25}$$

We wish to obtain information about the dynamics of the average market opinion

$$< n >_t := \sum_{n=-N}^{N} nP(n;t), \tag{4.26}$$

whose evolution through time is due to (4.25):

$$\begin{aligned}
<\dot{n}>_t : &= \frac{d}{dt} < n >_t = \sum_{n=-N}^{N} n\frac{d\,P(n;t)}{dt} \\
&= \sum_{n=-N+1}^{N} nw_{-+}(n-1)P(n-1;t) \;+ \sum_{n=-N}^{N-1} nw_{+-}(n+1)P(n+1;t) \\
&\quad - \sum_{n=-N}^{N-1} nw_{-+}(n)P(n;t) \;- \sum_{n=-N+1}^{N} nw_{+-}(n)P(n;t) \\
&= \sum_{n=-N}^{N} \left[w_{-+}(n) - w_{+-}(n)\right] P(n;t), \qquad (4.27)
\end{aligned}$$

where we have shifted the summation index in the first two terms by $-/+1$ and made use of the boundary conditions

$$w_{-+}(N) = w_{+-}(-N) = 0. \tag{4.28}$$

Solving (4.27) requires knowledge of the full probability distribution of n. It is however desirable to have an equation for the average market opinion that depends on mean values only. For that purpose an *opinion index* x,

$$x := n/N, \qquad -1 \leq x \leq 1, \tag{4.29}$$

is introduced, which in the limit $N \to \infty$ may be regarded as a continuous random variable with associated probability measure $P(x;t)$ normalized as

$$\int_{-1}^{1} P(x;t)\,dx \approx \sum_{x=-1}^{1} P(x;t)\Delta x = 1, \qquad \Delta x = \frac{\Delta n}{N} = \frac{1}{N}. \tag{4.30}$$

Abbreviating

$$K(x) := w_{-+}(x) - w_{+-}(x), \tag{4.31}$$

(4.27) may be reexpressed as

$$<\dot{x}>_t := \frac{d}{dt}\int_{-1}^{1} P(x;t)\,dx = \int_{-1}^{1} K(x)P(x;t)\,dx = <K(x)>_t. \tag{4.32}$$

If K is at least two times differentiable, it may be expanded in a Taylors series up to second order around $x = {<}x{>}_t$:

$$K(x) = K({<}x{>}_t) + K'({<}x{>}_t)(x - {<}x{>}_t) + \frac{1}{2}K''({<}x{>}_t)(x - {<}x{>}_t)^2 + \ldots, \tag{4.33}$$

which defining

$$\sigma^2(t) := {<}(x - {<}x{>}_t)^2{>}_t = {<}x^2{>}_t - {<}x{>}_t^2 \geq 0 \tag{4.34}$$

yields for the mean opinion index the approximate equation

$$\begin{aligned} {<}\dot{x}{>}_t &= K({<}x{>}_t) + \frac{1}{2}K''({<}x{>}_t)\sigma^2(t) + \ldots \\ &\approx K({<}x{>}_t), \end{aligned} \tag{4.35}$$

which is valid only for

$$|K({<}x{>}_t)| \gg \frac{1}{2}|K''({<}x{>}_t)\sigma^2(t)|. \tag{4.36}$$

Lux assumes therefore the probability distribution $P(x;t)$ to remain sharply peaked around its expected value $x = {<}x{>}_t$ at all times. If that is the case, the so called *quasi-meanvalue equation* (4.35) provides a tool to describe the dynamics of the mean opinion index in terms of a self-contained differential equation. That is, the change in the expected opinion index is a function of the expected opinion index alone, rather than its full probability distribution.[123]

Note that the closed self-contained dynamics of the mean opinion index ${<}x{>}_t$ hinges upon $\sigma^2(t)$ being small, i.e. single trajectories of x may not deviate substantially from their expected value. If such substantial deviations occur, the dynamics of ${<}x{>}_t$ is no longer correctly described by the quasi-meanvalue equation (4.35), nor is ${<}x{>}_t$ representative of individual trajectories of x.[124] However, it has been shown by Weidlich (2002: Chapter 12), that quasi-meanvalue equations such as (4.35) still characterize the mean evolution of any localized cluster of stochastic systems, even if that evolution belongs to a multimodal probability distribution. As such, the quasi-meanvalue equation (4.35) is even more appropriate to describe the mean evolution behaviour of

[123] Appendix A7 contains a derivation of quasi-meanvalue equations for the general case of arbitrarily many investment styles.

[124] For example, if the opinion index bifurcates into a multimodal probability distribution, its evolution is no longer meaningfully described by its unconditional expected value, as it lies somewhere between the states of maximal probability.

individual markets than the exact mean value equation (4.32), because it retains its interpretability as the equation of motion of a typical trajectory of x no matter whether $P(x;t)$ remains unimodal or not.

In order to capture the spirit of mimetic contagion, Lux models the individual transition probabilities $p_{\mp\pm}$ as

$$p_{-+}(x) = v\exp(\alpha x), \qquad p_{+-}(x) = v\exp(-\alpha x), \tag{4.37}$$

where α measures the strength of infection and v measures the speed of the infection process. Such a setup has the properties that *i)* both transition rates are positive definite, *ii)* p_{-+} and p_{+-} are symmetric with $dp_{\mp\pm}/p_{\mp\pm} = \pm\alpha\, dx$, and *iii)* $K(x)$ is infinitely differentiable. Note that v depends upon the time unit chosen in order to describe the dynamics of the opinion index x. Lux inserts these transition rates into the quasi-meanvalue equation (4.35) and provides a stability analysis of the resulting differential equation for the mean opinion index $<x>_t$. It turns out that the market has a unique stable equilibrium at $<x>_t = 0$ only for $\alpha \leq 1$. If the strength of infection parameter α becomes larger than that, the equilibrium of balanced opinions $<x>_t = 0$ becomes unstable and two symmetric stable bubble equilibria with a majority of either optimistic or pessimistic traders emerge.

In the next step the contagion process is linked with the price dynamics using the order based approach by Beja & Goldman (1980) and Day & Huang (1990). Each optimist/pessimist is assumed to demand $\pm t_N$ shares, such that the aggregate demand D_N of these noise traders becomes, making use of the definitions (4.17) and (4.29):

$$D_N = n_+ t_N - n_- t_N = xT_N, \qquad T_N := 2Nt_N. \tag{4.38}$$

Lux introduces then as a third trader group fundamentalists with aggregate excess demand D_F,

$$D_F = T_F(p_f - p), \qquad T_F > 0, \tag{4.39}$$

where p_f and p denote the fundamental value and the trading price respectively, and T_F stands for aggregate excess demand by fundamentalists per unit mispricing. A market maker adjusts the price proportional to the aggregate net asset demand of both trader types as in Beja & Goldman (1980):

$$\dot{p} := \frac{dp}{dt} = \beta(D_N + D_F) = \beta[xT_N + T_F(p_f - p)], \tag{4.40}$$

where β denotes the speed of price adjustment by the market maker. In order to allow for trend following (chartism) by the noise traders, Lux extends the individual transition probabilities $p_{\mp\pm}$ in (4.37) as

$$p_{-+}(x) = v\exp(\alpha_1 x + \alpha_2 \dot{p}/v), \qquad p_{+-}(x) = v\exp(-\alpha_1 x - \alpha_2 \dot{p}/v), \tag{4.41}$$

where v denotes the speed of opinion changes as before, but α has been split up into two parts: only α_1 describes the strength of contagion of other traders opinion, whereas α_2 constains the weight traders give to the current price trend when forming their opinion. The factor $1/v$ must be included with the current price trend $\dot{p}$ in order to make the transition rates independent of arbitrary changes in the time unit. Inserting these into the quasi-meanvalue equation (4.35) yields together with (4.40) a system of coupled differential equations for the mean opinion index $<x>_t$ and the trading price p, which as in the pure contagion case with transition rates (4.37) have a unique equilibrium at $<x>_t = 0$ and $p = p_f$ for $\alpha_1 \leq 1$, but two additional symmetric bubble equilibria at $p \neq p_f$ otherwise. These additional bubble equilibria become the further displaced from the fundamental price the larger the noise trader demand paramer T_N is relative to the fundamentalist demand parameter T_F, and with increasing infection parameter α_1. The fundamental equilibrium at $p = p_f$ is no longer necessary stable for $\alpha_1 \leq 1$, but requires both infection parameters α_1 and α_2, the speed of the infection process v, and the noise trader demand parameter T_N to be sufficiently small compared to the fundamental demand parameter T_F. If the fundamental equilibrium is unique but repelling, then at least one stable limit cycle exists such that all trajectories of the system converge to a periodic orbit in the $(<x>_t, p)$ space. The model therefore explains periodic switching between over- and undervaluation by means of an endogenous process of mimetic contagion and trend chasing by noise traders.

4.3 The Model by Lux and Marchesi

In this section I shall not only review but also replicate the results of a simulation study by Lux & Marchesi (2000) which replicated the main stylized facts of financial return series. I lay so much emphasis on this model, because the model to be developed in chapter 5 may well be regarded as a multivariate extension of the univariate setup by Lux & Marchesi. Because the modelling technique of the setup by Lux & Marchesi and my own is very similar, replicating their results serves as a valuable tool for cross checking the robustness of the computer program designed for modelling my own specification.

Furthermore, the simulation study by Lux & Marchesi (2000) will serve as an example to demonstrate the before mentioned inherent weakness of the order-based setup by Beja & Goldman (1980) of producing integrated trader inventories. As it turns out the model is extremely successful in replicating the main stylized facts of financial returns, that is uncorrelated returns with clustered, long range dependent volatility and heavy tails with a realistic tail index, however at the unacceptable cost of generating unbounded traders holdings, which have not been explored in the original simulation study by Lux & Marchesi.

As a minor point it will be noted that the simulated price series in Lux & Marchesi (2000) do not have the unit root property as claimed by the authors, but were probably the result of an incorrect application of the Dickey-Fuller test.[125] This is only of minor importance because the failure of the simulations to produce integrated price series is only due to the simplifying assumption of a constant fundamental value, which is easily healed by letting the fundamental price follow an integrated process. Lux & Marchesi (1999) show that assuming the intrinsic value to follow geometric Brownian motion leads to integrated prices with just as realistic return properties as those in Lux & Marchesi (2000). As such, the simulated time series in Lux & Marchesi (2000) should be thought of as deviations from the fundamental price rather than prices themselves, illustrating the essential content of the interacting agent hypothesis: that many of the stylized facts may be explained from the interaction of market participants alone, without resorting into some unobservable news generating process. The unit root property of financial prices would be exempt from such a behavioral explanation and instead be regarded as a natural consequence of the unit root property of intrinsic values.

[125]Thomas Lux agrees with this interpretation (personal communication).

4.3.1 The Model

The simulation studies by Lux & Marchesi (1999, 2000) are based upon a model by Lux (1998), which we in the following briefly sketch. The basic setup is similar to Lux (1995) with the additional feature that this time agents are allowed to switch also between the fundamentalist and noise trader (in the following denoted as chartist) subgroup.

In the market there are N speculators which may be subdivided into charists n_c and fundamentalists n_f according to

$$n_c + n_f = N. \tag{4.42}$$

As before, the chartists may be further subdivided into n_+ optimists and n_- pessimists:

$$n_+ + n_- = n_c. \tag{4.43}$$

An opinion index x is introduced similar to (4.29):

$$x := \frac{n_+ - n_-}{n_c}, \qquad -1 \leq x \leq 1. \tag{4.44}$$

The fraction of chartists in the market is denoted by z:

$$z := \frac{n_c}{N}, \qquad 0 \leq x \leq 1. \tag{4.45}$$

The opinion formation process within the chartist subgroup is modelled using the transition rates (4.41) reinterpreted as being conditional upon an interaction with another chartist taking place, which is assumed to happen with an unconditional probability given by the relative frequency of chartists in the market $z = n_c/N$. The unconditional transition rates of an individual chartist to move from the pessimist to the optimist subgroup p_{-+} and vice versa p_{+-} are therefore given by

$$p_{\mp\pm} = v_1 \left(\frac{n_c}{N} \exp(\pm U_1) \right), \qquad U_1 = \alpha_1 x + \alpha_2 \frac{\dot{p}}{v_1} \tag{4.46}$$

using the same notation as in (4.41) except that the speed of opinion revaluation parameter v has been renamed into v_1 in order to distinguish it from the speed of contagion between the chartist and fundamentalist subgroup to be discussed below.

Switches between chartists and fundamentalists are driven by expected or realized excess profits above the real rate of the economy R, which is assumed to equal the return of the risky asset in the case of a constant trading price p at fundamental value p_f. That is $R = r/p_f$, where r denotes the dividend of the stock. In that case the expected excess returns by fundamentalists are given by $s|(p_f - p)/p|$, where s is a discount factor, since fundamentalists profits will first occur in the future when the trading price will have returned to its fundamental value.

Bullish chartists, who invest into the risky security, receive its nominal dividend r and the price change $\dot{p}/v_2$, but forego the average rate of return of the economy R, such that their excess return is $(r+\dot{p}/v_2)/p-R$. The utilities of moving from the fundamentalist to the optimist subgroup $U_{2,+}$ and vice versa $-U_{2,+}$ are therefore given by

$$U_{2,+} = \alpha_3 \left(\left(\frac{r+\dot{p}/v_2}{p} - R \right) - s \left| \frac{p_f - p}{p} \right| \right), \tag{4.47a}$$

where α_3 measures the sensitivity of traders to differences in profits. Bearish chartists on the other hand, who short the risky asset in order to invest into the overall economy receive $R-(r+\dot{p}/v_2)/p$. The utilities of moving from the fundamentalist to the pessimist subgroup $U_{2,-}$ and to move from the pessimist to the fundamentalist subgroup $-U_{2,-}$ are therefore

$$U_{2,-} = \alpha_3 \left(\left(R - \frac{r+\dot{p}/v_2}{p} \right) - s \left| \frac{p_f - p}{p} \right| \right). \tag{4.47b}$$

Taking into account the probability of interaction of the relevant trader subgroups as measured by their relative frequency yields for the transition rates from fundamentalists to the two kind of chartists $p_{f+/-}$ and vice versa $p_{+/-f}$ in analogy to (4.46):

$$p_{f+} = v_2 \frac{n_+}{N} \exp(U_{2,+}), \qquad p_{+f} = v_2 \frac{n_f}{N} \exp(-U_{2,+}), \tag{4.48a}$$

$$p_{f-} = v_2 \frac{n_-}{N} \exp(U_{2,-}), \qquad p_{-f} = v_2 \frac{n_f}{N} \exp(-U_{2,-}). \tag{4.48b}$$

The quasi-meanvalue equations for n_+ and n_- read now[126]

$$\dot{n}_+ = n_- p_{-+} + n_f p_{f+} - n_+(p_{+-} + p_{+f}), \tag{4.49a}$$

$$\dot{n}_- = n_+ p_{+-} + n_f p_{f-} - n_-(p_{-+} + p_{-f}), \tag{4.49b}$$

[126] This is the case of $L = 3$ investment styles in equation (A7.14).

where we have dropped the brackets $<>_t$ to indicate the expectation operator at time t for notational convenience. The equations of motion for n_c and n_f may be inferred from the defining equations (4.42) and (4.43).

The link with the price dynamics is as in Lux (1995) given by

$$\dot{p} = \beta ED = \beta(ED_c + ED_f) = \beta[(n_+ - n_-)t_c + n_f t_f(p_f - p)], \tag{4.50}$$

where β denotes again the reaction speed of the market maker, ED is the aggregate excess demand of both fundamentalists ($ED_f = n_f t_f(p_f - p)$) and chartists ($ED_c = (n_+ - n_-)t_c$), and t_c and t_f denote the number of shares traded by single chartists and fundamentalists respectively.

Lux shows that the quasi-meanvalue equations (4.49) and the price equation (4.50) may be transformed into the following system of coupled differential equations for the state variables x, z and p:

$$\begin{aligned} \dot{x} &= 2zv_1[\tanh(U_1) - x]\cosh(U_1) + (1-z)(1-x^2)v_2[\sinh(U_{2,+}) - \sinh(U_{2,-})], \\ \dot{z} &= (1-z)zv_2[(1+x)\sinh(U_{2,+}) + (1-x)\sinh(U_{2,-})], \\ \dot{p} &= \beta[xzT_c + (1-z)(p_f - p)T_f], \qquad \text{with } T_c := Nt_c \text{ and } T_f := Nt_f. \end{aligned} \tag{4.51}$$

It turns out that the only stationary solutions of (4.51) are given by:

(i) *x^*=0, $p^* = p_f$ with arbitrary z,*

(ii) *x^*=0, z^*=1 with arbitrary p,*

(iii) *z^*=0, $p^* = p_f$ with arbitrary x.*

The last two equilibria describe absorbing states in which either the group of chartists or fundamentalists has declined to zero. The interest of Lux & Marchesi is in equilibria of the first type, which are characterized by efficient price formation and balanced disposition of opinion among chartists, implying that neither fundamentalists nor chartists have an advantage due to vanishing utilities $U_{2,+}$ and $U_{2,-}$ in (4.47).[127] Lux derives the

[127] recall $R = r/p_f$.

following necessary conditions for stability of equilibria of the first type along the line $(x^* = 0, p^* = p_f, z^*)$:

$$1. \qquad 2z^*v_1\left(\alpha_1 + \alpha_2\frac{\beta}{v_1}z^*T_c - 1\right) + 2(1-z^*)\alpha_3\beta z^*\frac{T_c}{p_f} - \beta(1-z^*)T_f < 0, \tag{4.52a}$$

$$2. \qquad \alpha_1 < 1 + \alpha_3\frac{v_2T_cR}{v_1T_fp_f}. \tag{4.52b}$$

Given that the second condition is fulfilled, the first condition implies that there is an upper threshold fraction of chartists in the market $z_{\max}$ beyond which equilibria of the first kind become unstable. Solving (4.52a) yields

$$\begin{aligned} z_{\max} &= \sqrt{\left(\frac{b}{2a}\right)^2 + \frac{\beta T_f}{a}} - \frac{b}{2a} \qquad \text{with} \\ a &:= 2\beta T_c\left(\alpha_2 - \frac{\alpha_3}{p_f}\right) \qquad \text{and} \\ b &:= 2v_1(\alpha_1 - 1) + \beta\left(2\alpha_3\frac{T_c}{p_f} + T_f\right). \end{aligned} \tag{4.53}$$

In the remainder of this section we shall follow Lux & Marchesi (2000) in simulating the stochastic system underlying (4.51) around equilibria of the first type with the fraction of chartists in the market obeying $z < z_{\max}$.

4.3.2 Simulation Study

The system of differential equations (4.51) describes the coupled population and price dynamcis as a process in continuous time. As such it can only be approximated in computer simulations. Lux & Marchesi choose to split each integer time step into 100 microsteps of equal length $\Delta t = 0.01$, at each of which the composition of the trader population may change according to the transition rates given in (4.46) and (4.48). Note that because these transition rates describe the probability of a population change *per unit time*, in the actual simulations they have to be divided by the number of micro time steps in order to yield the transition probability during the time interval Δt. Lux & Marchesi note that during periods of high volatility it was necessary to increase the precision of the simulations by a factor 5 to $\Delta t = 0.002$. Because computation speed is no longer such a serious constraint as it was during the time of the simulation study by Lux & Marchesi, I shall use this higher precision of 500 micro steps per unit time interval throughout.

Furthermore, in the system of differential equations (4.51) the price p is the only non-stochastic variable in the sense that it does not describe the expected value of a stochastic process but has been arrived at in a purely deterministic manner according to equation (4.50) (albeit with inputs derived from a stochastic process). Lux & Marchesi wish to generate p in an anologous manner to x and z by formulating the following stochastic process with expected time change $<\dot{p}>_t$ given by (4.50): They split the price unit (1 dollar, say) into 100 elementary units (cents) and consider the probability of the price to move from one elementary unit to the next within Δt. For that purpose, a small noise term $\mu \sim \mathcal{N}(0, \sigma^2)$ is added to the excess demand ED_t at time t and the transition probabilities to move one cent up (π_{p+}) or down (π_{p-}) during the time interval Δt are modelled as

$$\pi_{p+} = 100 \max[0, \beta(ED_t + \mu)]\Delta t, \qquad \pi_{p-} = -100 \min[0, \beta(ED_t + \mu)]\Delta t, \tag{4.54}$$

such that the expected price change $<\Delta p>_t$ between t and $t + \Delta t$ becomes

$$<\Delta p>_t = 0.01 <\pi_{p+}>_t - 0.01 <\pi_{p-}>_t = \beta ED_t \Delta t \tag{4.55}$$

and (4.50) may be interpreted as

$$<\dot{p}>_t = \beta ED_t. \tag{4.56}$$

The binary price adjustment rule (4.54) leaves only the possibilities $\Delta p = -0.01$, 0, and $+0.01$ as possible inputs for $\dot{p} \approx \Delta p / \Delta t$ as an approximation for the time derivative of p in the equations of motion (4.51) from the simulated prices between t and $t - \Delta t$. I follow Lux & Marchesi in calculating $\dot{p}$ from the longer time interval $[t - 0.2, t)$ in order to allow for a broader set of values. I also follow them in setting a lower bound of 4 out of $N = 500$ agents in any trader subpopulation in order to avoid occurence of the absorbing states $z = 0$ and $z = 1$ in the simulations, that is the stationary equilibria of type (ii) and (iii) in the differential equation system (4.51).

The simulations run then as follows. Initially, the trading price is set to its fundamental value $p = p_f$, and the traders are randomly distributed over the subpopulations n_+,

Table 1. Parameter sets used by Lux & Marchesi (2000) and in the replicating simulations of this section.

	Parameter set I	Parameter set II	Parameter set III	Parameter set IV
N	500	500	500	500
p_f	10	10	10	10
r	0.004	0.004	0.004	0.004
v_1	3	4	0.5	2
v_2	2	1	0.5	0.6
β	6	4	2	4
T_c	10	7.5	10	5
T_f	5	5	10	5
α_1	0.6	0.9	0.75	0.8
α_2	0.2	0.25	0.25	0.2
α_3	0.5	1	0.75	1
s	0.75	0.75	0.8	0.75
σ	0.05	0.1	0.1	0.05

n_- and n_f in such a manner that the stability condition $z < z_{\max}$ holds. Traders are then allowed to change their strategy according to the transition probabilities (4.46) and (4.48) and subsequently the trading price is adjusted according to the transition probabilities (4.54), using the parameters given in table 1. The matlab code for running the simulations can be found in section A1 of the appendix.

Lux & Marchesi implement the strategy switches of the traders by drawing for each trader a uniform random number on the interval [0,1] and comparing it with the relevant transition probability. That is, they generate binomially distributed numbers of randomly switching agents as sums of independent Bernoulli distributed random variables with the relevant transition probability. The statistics toolbox for use with matlab contains a generator of binomially distributed pseudo-random numbers based upon summation of independent Bernoulli distributed random variates (command: `binornd`), which is unfortunately quite slow due to its inefficient coding. A much faster way to generate a pseudo-random number `k` from a binomial distribution with parameters `n` and `p` based upon the same idea is given by the following simple command:

```
k = sum(repmat(p,n,1)>rand(n,1)),
```

which fills a `n`×1 column vector with `p`, compares it element wise to a `n`×1 column vector filled with $\mathcal{U}[0,1]$ distributed pseudo-random numbers, and sums up the number of occurences of p exceeding its associated random number.

The execution speed is further improved by generating random variates of agents leaving from the same subpopulation within the same matrix. The binomial draws of agents leaving their strategy in lines 170 to 173 of appendix A1 were therefore originally coded as

```
170    %(1*2) binomial draws of agents leaving their strategy
171    npout = sum(repmat([ppm(2) pcf(1,2)],np,1)>rand(np,2));
172    nmout = sum(repmat([ppm(1) pcf(2,2)],nm,1)>rand(nm,2));
173    nfout = sum(repmat([pcf(1,1) pcf(2,1)],nf,1)>rand(nf,2));
```

which does the same as before on two-columned matrices with the number of rows given by the number of agents in the relevant subpopulation and the columns filled with the relevant transition probabilities.

Generating binomially distributed random variates from summing up Bernoulli random numbers is however inefficient in our case of large n and small p due to the many calls of the random number generator. An algorithm for producing binomially distributed random variates with only a single call of the uniform random number generator and n*p expected loops is given by the BINV algorithm described in Kachitvichyanukul & Schmeiser (1988) and implemented under the name fastbin in lines 288 to 312 of appendix A1. The BINV algorithm uses the inversion method for transforming $\mathcal{U}[0,1]$ distributed random variates into a random number with distribution function F. That is, defining the generalized inverse of a function F on [0,1], F^{-}, as

$$F^{-}(u) := \inf\{x;\, F(x) \geq u\},$$

then $F^{-}(U)$ will have the distribution F, if $U \sim \mathcal{U}[0,1]$. The BINV algorithms exploits the recursive formula

$$f_B(k) = f_B(k-1)\frac{n-k+1}{k}\frac{p}{1-p} \quad \text{for } k = 1, 2, \ldots, n \tag{4.57}$$

of the binomial distribution

$$f_B(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, 2, \ldots, n \tag{4.58}$$

in stepwise searching for the generalized inverse of a $\mathcal{U}[0,1]$ distributed pseudo-random number starting from $k = 0$. Because the expected value of k is $n \cdot p$, there will on average only $n \cdot p$ steps be needed in determination of k.

Having a fast binomial random number generator is important because the program spends most of its execution time on calculating the number of agents changing their strategy. I found the BINV algorithm first in the literature after I had already implemented a preliminary version of the algorithm in appendix A1 based upon summation of independent Bernoulli random variables. This provides us with two independent sets of simulations at least for Lux' state variables p, x and z, in the following referred to as Simulation I (Bernoulli rv's) and Simulation II (BINV algorithm), where applicable. In the remainder of this section I shall follow Lux & Marchesi in generating 20,000 observation points for each of the four parameter sets and apply the same battery of tests to them as they did.

Consider first the simulated return series over 20,000 observations in figures 1 and 2. All time series exhibit sudden outbreaks of volatility similar to what is empirically observed in financial markets.[128] Comparison with the plots of the chartist index z in figures 3 and 4 reveals that the volatility outbursts are related to the number of chartists. Volatility clusters are always accompanied by above average presence of chartists in the market. Lux & Marchesi attribute this to self-reinforcing trends under dominance of chartists, which become quickly reversed once large enough price deviations from fundamental value create sufficient profit opportunities for fundamentalists to act as a counterforce against excessive mispricing. Note the occasional occurence of $z > z_{\max}$, where the market is expected to loose its stability, in parameter sets I and III of simulation I (first and third panel in figure 3). The same does not happen in simulation II, because there occurence of $z > z_{\max}$ has been artificially prevented in the block from line 194 to 201 of the code in appendix A1, which had not yet been implemented in the preliminary version used in simulation I. Originally z was allowed to exceed $z_{\max}$ in the simulation runs II as well, but this lead to simulation breakdowns due to exploding price oscillations in accordance with the stability condition $z < z_{\max}$. The occasional violation of this stability condition, which may or may not lead to market instability,

[128] see section 2.5.

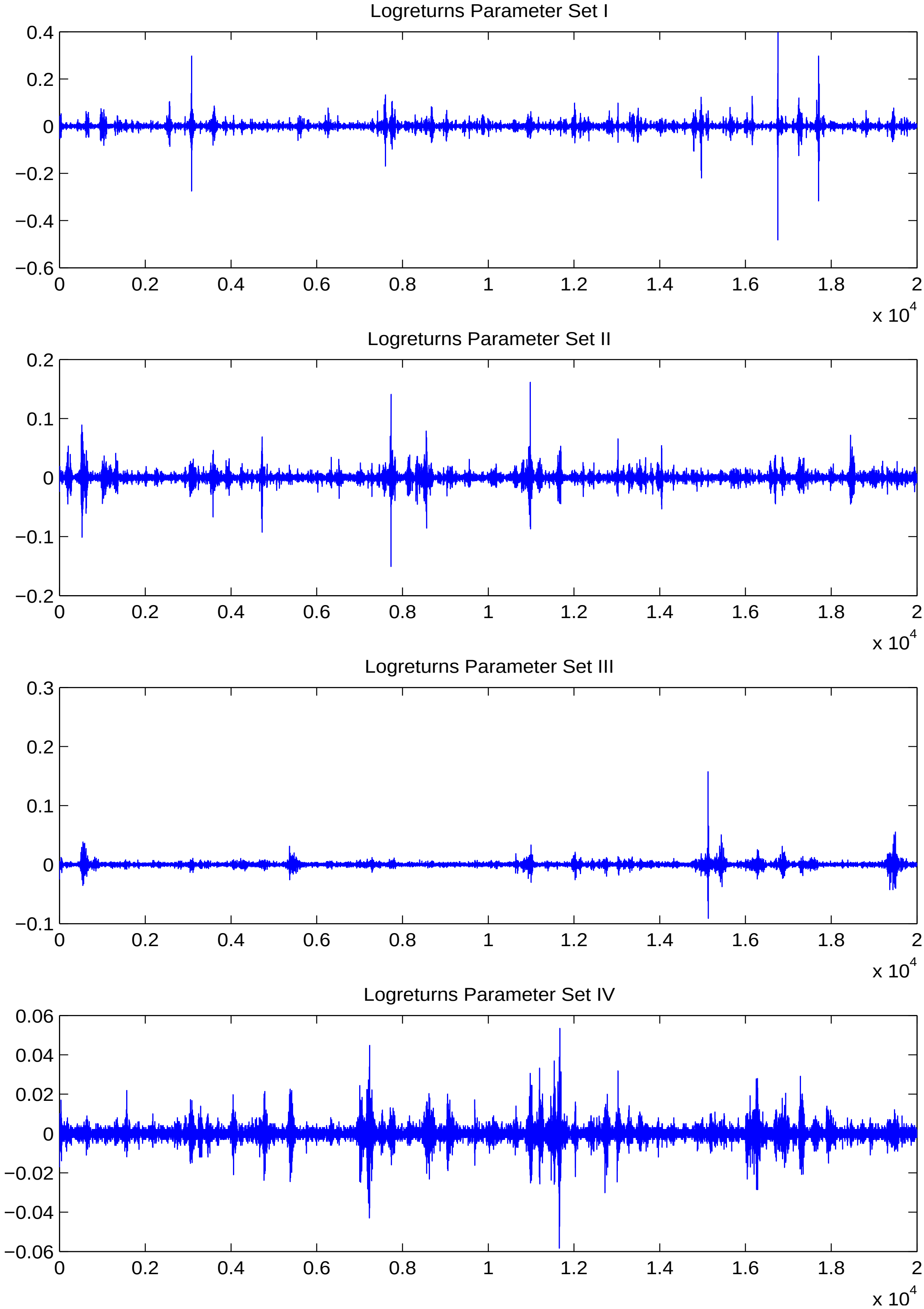

Figure 1. Logreturns over 20,000 integer time steps (Simulation I).

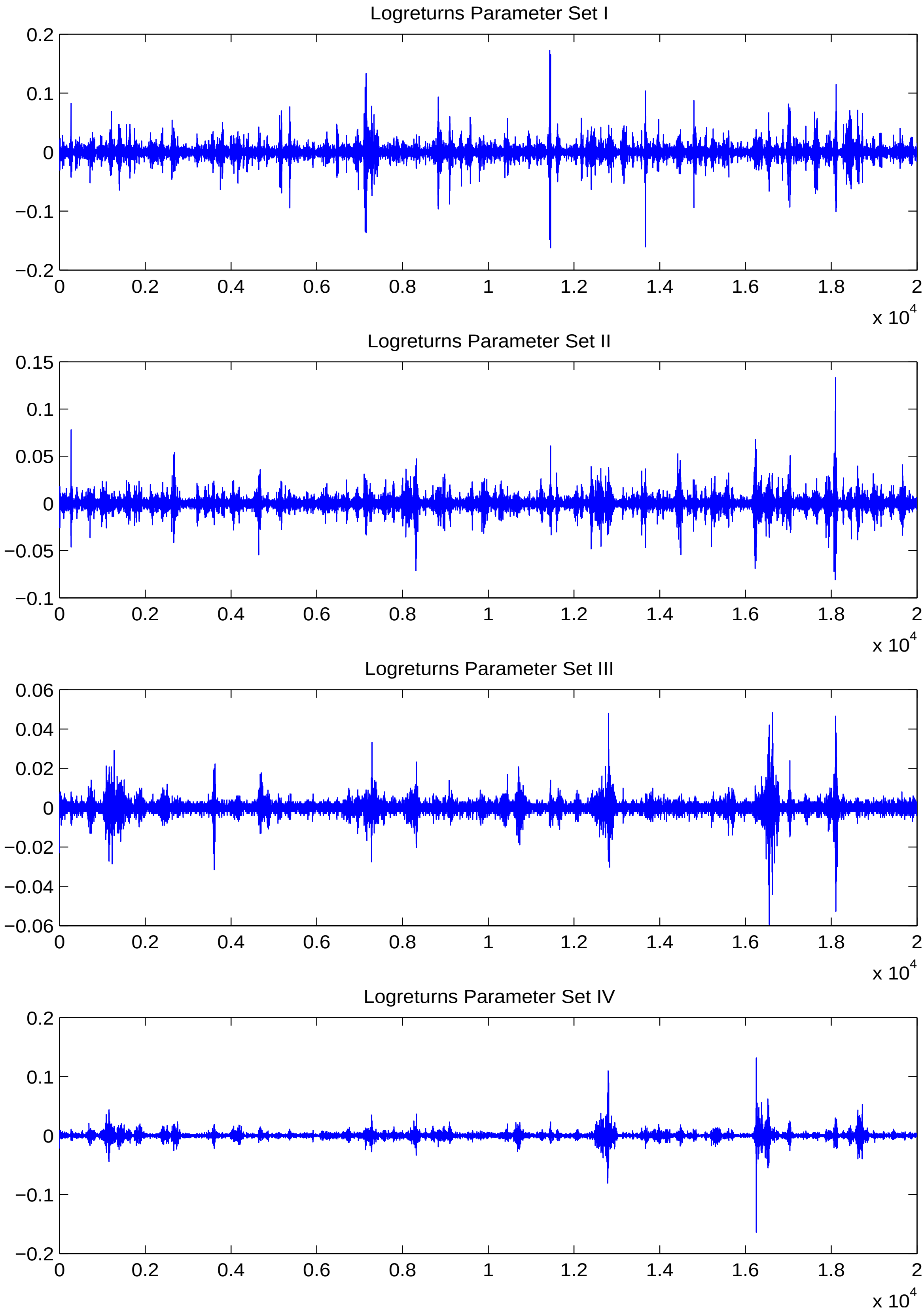

Figure 2. Logreturns over 20,000 integer time steps (Simulation II).

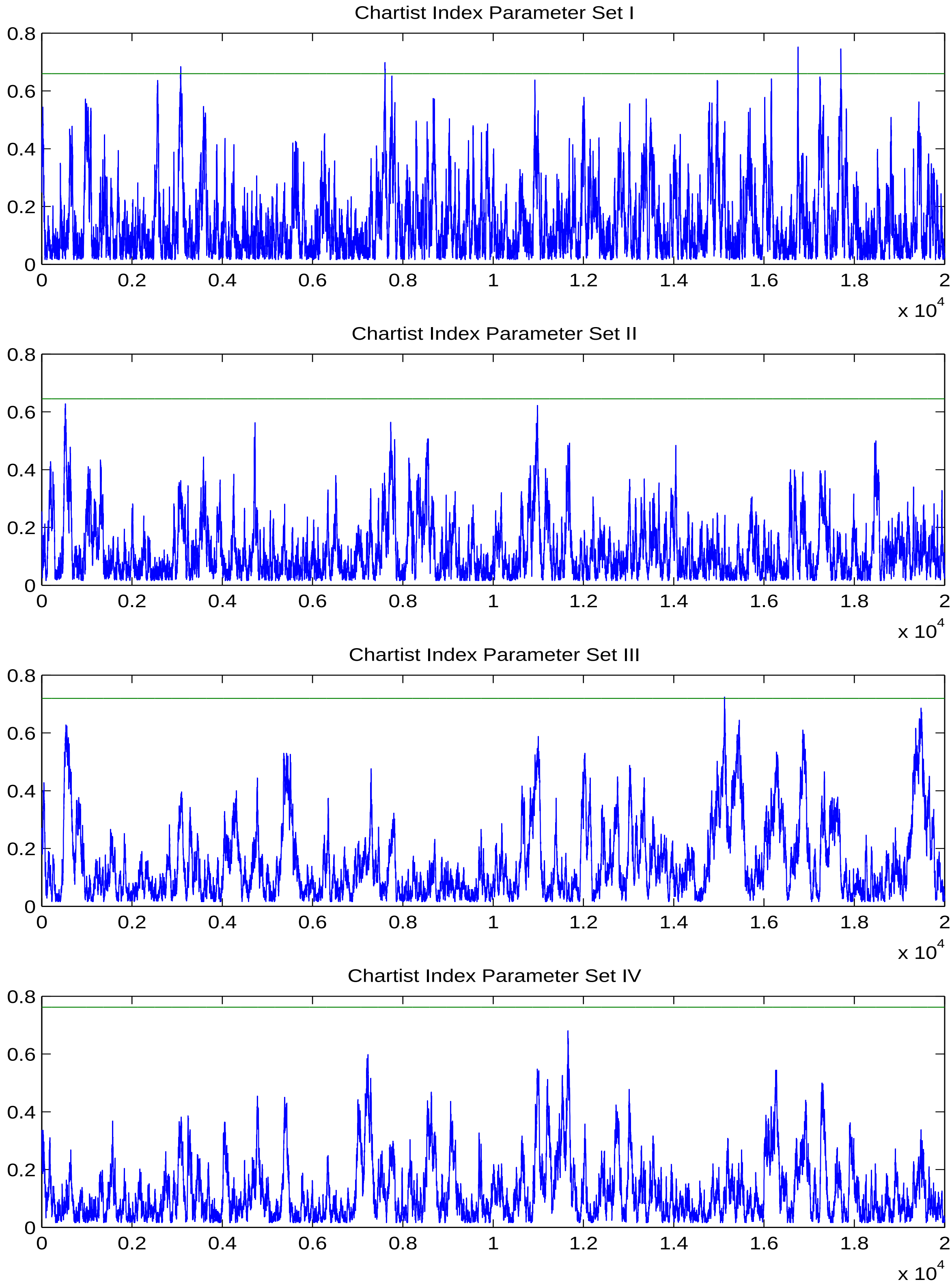

Figure 3. Chartist index z over 20,000 integer time steps (Simulation I). The horizontal lines indicate the threshold $z_{\max}$, beyond which the first type equilibrium of (4.51) is expected to loose its stability.

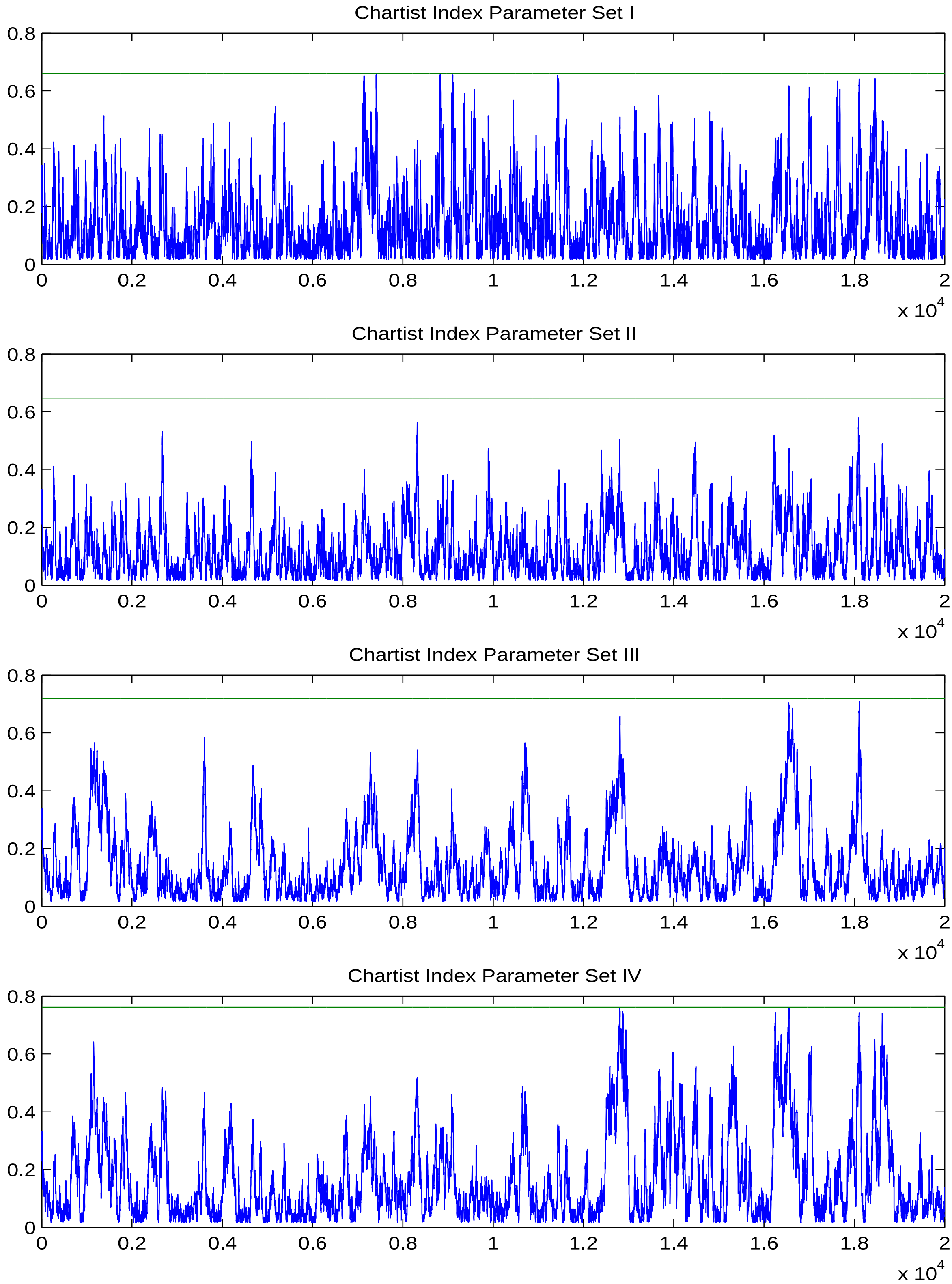

Figure 4. Chartist index z over 20,000 integer time steps (Simulation II). The horizontal lines indicate the threshold $z_{\max}$, beyond which the first type equilibrium of (4.51) is expected to loose its stability.

Table 2. Median estimates of the tail index over ten samples of 2,000 observations each and the range of estimates for common choices of the tail region.

Parameter Set I	min	2.5%tail median	max	min	5%tail median	max	min	10%tail median	max
Simulation I	1.86	3.34	4.80	1.60	2.57	3.15	1.58	2.27	2.66
Simulation II	2.12	3.23	3.59	1.97	2.51	3.56	1.73	2.17	2.65
Lux&Marchesi	1.61	2.04	4.50	1.51	2.11	2.64	1.26	1.93	2.44
Parameter Set II	**min**	**2.5%tail median**	**max**	**min**	**5%tail median**	**max**	**min**	**10%tail median**	**max**
Simulation I	2.18	2.61	4.44	2.21	2.32	3.09	1.76	2.19	2.67
Simulation II	2.34	3.43	4.08	2.12	2.96	3.46	2.11	2.35	2.95
Lux&Marchesi	2.28	2.82	3.73	2.00	2.52	3.17	1.55	2.18	2.36
Parameter Set III	**min**	**2.5%tail median**	**max**	**min**	**5%tail median**	**max**	**min**	**10%tail median**	**max**
Simulation I	2.02	3.34	4.58	1.81	2.93	4.43	1.46	2.28	5.09
Simulation II	1.58	3.21	4.81	1.25	2.81	4.12	1.72	2.67	3.33
Lux&Marchesi	2.41	4.63	6.82	2.33	3.48	8.60	1.80	2.86	4.84
Parameter Set IV	**min**	**2.5%tail median**	**max**	**min**	**5%tail median**	**max**	**min**	**10%tail median**	**max**
Simulation I	2.92	3.70	7.75	1.97	2.90	4.13	1.94	2.83	3.85
Simulation II	2.07	3.25	5.06	1.95	2.98	3.74	1.65	2.33	3.24
Lux&Marchesi	2.11	3.08	4.06	2.13	2.46	7.68	1.65	1.97	3.18

illustrates the difference between the stochastic dynamics of the state variables x, p and z, and the dynamics of their expected values $<\!x\!>_t$, $<\!p\!>_t$ and $<\!z\!>_t$, whose dynamics is described by the deterministic differential equations (4.51). If (4.51) would describe the dynamics of the state variables themselves, the case $z > z_{\max}$ could not occur as long as z was initialized below this threshold. On the other hand, occurence of $z > z_{\max}$ would necessarily lead to market instability. Occasional violation of the stability condition $z < z_{\max}$ with spontaneous return to stability is only possible because the state variables x, p and z fluctuate stochastically around their expected values $<\!x\!>_t$, $<\!p\!>_t$ and $<\!z\!>_t$.

I shall now turn to the statistical analysis of the simulated return series starting with the fat-tailedness of the unconditional return distribution. Consider for that purpose the Hill estimates of the tail index α[129] in table 2 generated with the matlab code

[129] see section 2.4.

Table 3. Kurtosis estimates over the full sample of 20,000 observations.

	Simulation I	Simulation II	Lux&Marchesi
Parameter Set I	147.47	27.59	135.73
Parameter Set II	43.61	23.46	16.10
Parameter Set III	147.95	34.65	27.11
Parameter Set IV	22.51	96.44	37.74

presented in appendix A2. Making use of the symmetry of returns, the positive and negative tails have been merged into absolute returns in order to provide a better statistical basis for the estimation by means of a larger sample size. All return series have been seperated into 10 subsamples of 2,000 observations each, in order to facilitate comparison with the simulation results by Lux & Marchesi (2000). The table reports the smallest, median, and highest tail index estimate within the ten subsamples for the commonly chosen tail regions of the largest 2.5%, 5%, and 10% absolute returns. The results agree both with Lux & Marchesi and the empirical findings in financial markets in that the median estimates hoover around in the range 2 to 4 with decreasing α-estimate for increasing tail size.

A less precise measure of fat-tailedness is given by the sample kurtosis,[130] presented in table 3. Here the agreement within the simulations and with Lux & Marchesi is only qualitative in that all simulations generate at least double digit estimates, indicating strong fat-tailedness. The numerical differences of the estimates between the different simulation runs are however often large. This is not surprising, given that tail indices below 4 imply that the kurtosis of the process is not defined.[131] We may therefore not expect the sample kurtosis to converge to any specific number in such processes. This is illustrated in figure 5, which shows the kurtosis estimate of the unconditional return distribution as a function of increasing sample size. It is seen that the kurtosis estimate contains sudden jumps after which—even though initially leveling off—it does not converge to any stationary level. This finding is in harmony with the behaviour of the kurtosis estimator in empirical financial data.[132] Comparison with the return series in figures 1 and 2 reveals that the sudden jumps in the kurtosis estimate are caused by

[130] see section 2.3.
[131] see section 2.4.
[132] see e.g. Cont (2001).

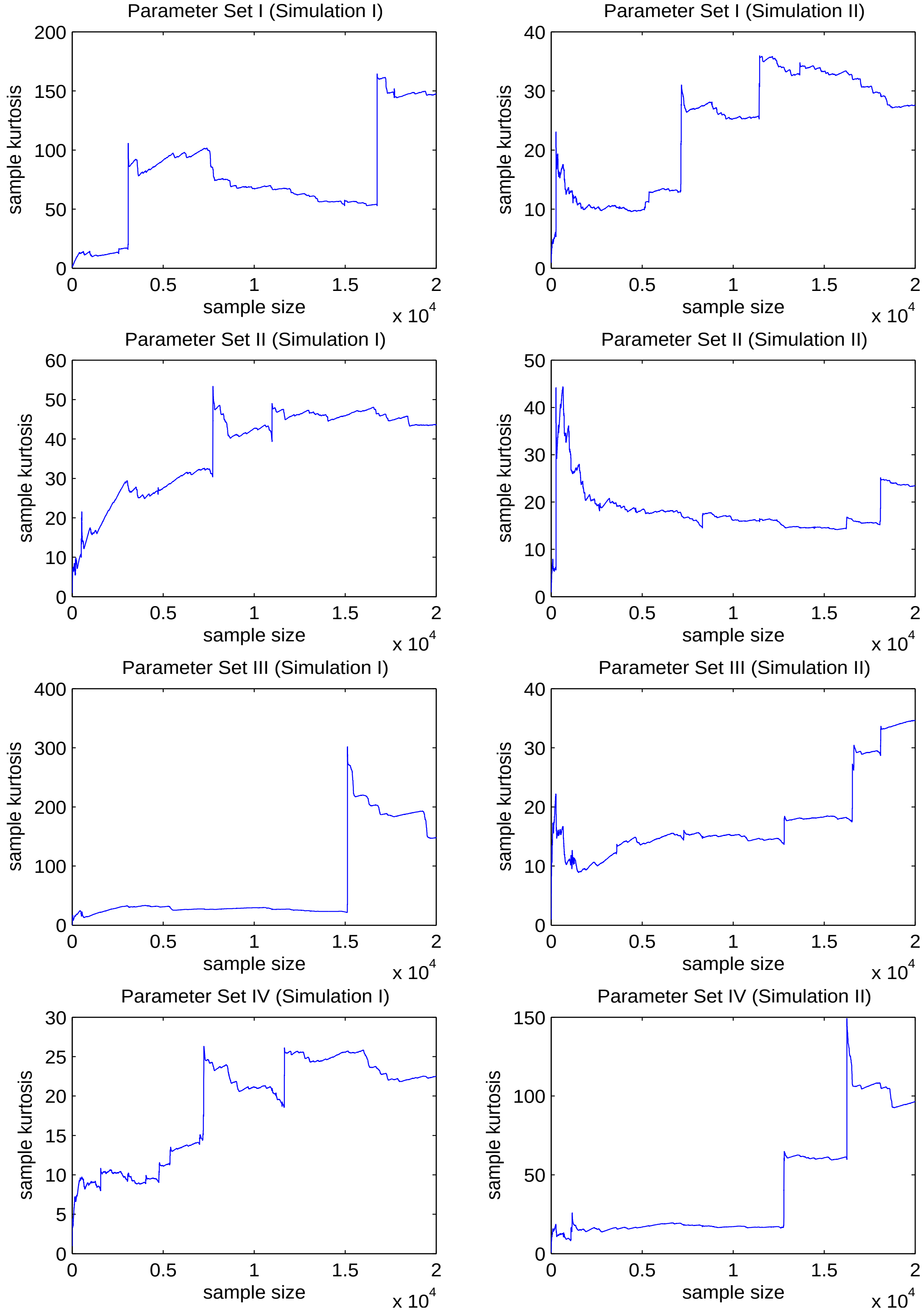

Figure 5. Sample kurtosis estimates for increasing sample size.

the most extreme return observations relative to their history, as is to be expected from the definition of kurtosis (2.4). This estimate initially levels off, as more observations accumulate, but for tail indices smaller than 4, even more extreme observations are generated before the estimator reaches a staionary value.

Consider next the issue of long range dependence in volatility. Figure 6 shows the autocorrelation functions of raw, squared, and absolute returns for up to 300 lags. The autocorrelations for raw returns fluctuate around zero in accordance with the empirical findings discussed in section 2.2. The slowly decaying autocorrelation functions of absolute returns however, point at the possibility of long memory in volatility, in particular for parameter sets III and IV. The autocorrelation function of squared returns remains also positive for a large number of lags in most simulations, but decays somewhat faster than for absolute returns, in harmony with empirical findings.[133] On the other hand, the autocorrelation function of squared returns appears to decay somewhat too fast to be supportive of long range dependent volatility, in particular in parameter sets I and II. Lux & Marchesi provide autocorrelation diagrams only for parameter set IV, which look similar to those presented here for parameter sets III and IV with slowly decaying autocorrelation function for squared and particularly absolute returns.

In order to test formally for long range dependence in volatility, tables 4 to 7 contain the results of logperiodogram regressions for estimation of the long memory parameter d in squared and absolute returns using the algorithm presented in appendix A3. Lux & Marchesi divide the full sample of 20,000 observations into 10 subsamples of 2,000 observations, estimate d in each of these subsamples, and report the median of these estimates together with its range and the number of significantly positive d-estimates at 5% level. I have additionally included the d-estimate over the full sample. Looking at these first, it turns out that evidence for long memory in absolute returns with significantly positive $\hat{d}$ is provided only for parameter sets III and IV in harmony with the autocorrelation diagrams in figure 6. Furthermore, only in simulation II of parameter set III the decay in the autocorrelation of squared returns is slow enough to provide evidence of long memory. All other estimates of d for squared and absolute returns over the full sample are positive, but insignificantly so. Turning to the estimation results in

[133]see e.g. Ding et al. (1993) and the discussion in section 2.7.

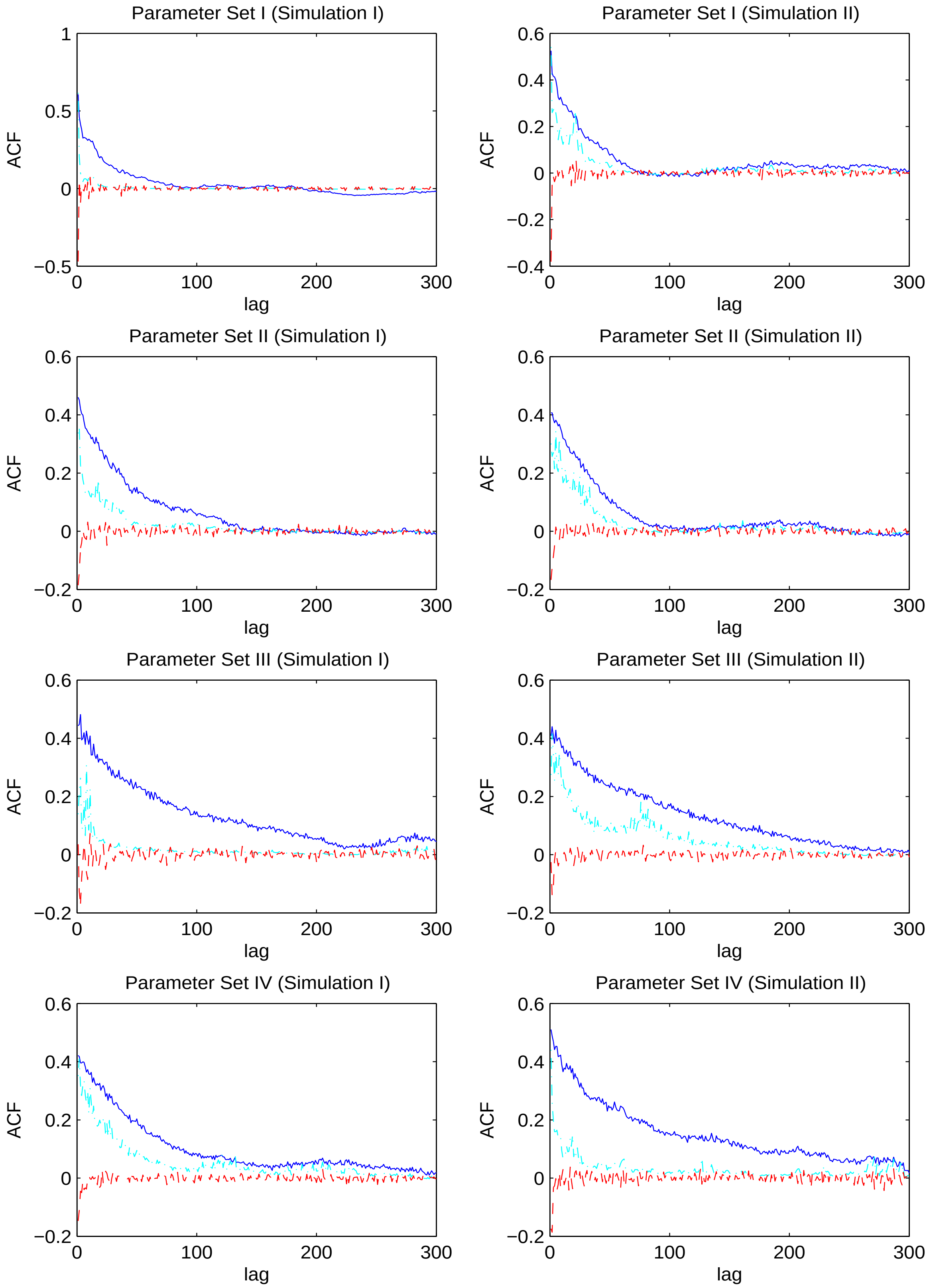

Figure 6. Autocorrelation diagrams of absolute (upper solid line), squared (middle dashed line) and raw returns (lower dashed line) over 300 lags.

Table 4. Estimates of the long memory parameter d for squared returns (upper panel) and absolute returns (lower panel) over the full sample of 20,000 observations and over 10 subsamples of 2,000 observations each for parameter set I. The last column contains the number of significantly positive estimated long memory parameters within the ten subsamples at a significance level of 5%. Significantly positive estimates of d over the full sample are marked with an asterix (*).

Squared Returns	$\hat{d}$ full sample	$\min(\hat{d})$	10 samples median$(\hat{d})$	$\max(\hat{d})$	# $\hat{d}$ sign. > 0 in 10 samples
Simulation I	0.07	-0.01	0.31	0.50	2
Simulation II	0.09	0.08	0.24	0.52	2
Lux&Marchesi		0.06	0.17	0.56	4
Absolute Returns	**$\hat{d}$ full sample**	**$\min(\hat{d})$**	**10 samples median$(\hat{d})$**	**$\max(\hat{d})$**	**# $\hat{d}$ sign. > 0 in 10 samples**
Simulation I	0.15	0.21	0.39	0.49	3
Simulation II	0.13	0.22	0.39	0.54	4
Lux&Marchesi		0.21	0.38	0.64	8

Table 5. Estimates of the long memory parameter d for squared returns (upper panel) and absolute returns (lower panel) over the full sample of 20,000 observations and over 10 subsamples of 2,000 observations each for parameter set II. The last column contains the number of significantly positive estimated long memory parameters within the ten subsamples at a significance level of 5%. Significantly positive estimates of d over the full sample are marked with an asterix (*).

Squared Returns	$\hat{d}$ full sample	$\min(\hat{d})$	10 samples median$(\hat{d})$	$\max(\hat{d})$	# $\hat{d}$ sign. > 0 in 10 samples
Simulation I	0.12	0.13	0.32	0.65	1
Simulation II	0.06	0.06	0.39	0.55	3
Lux&Marchesi		0.37	0.54	0.86	10
Absolute Returns	**$\hat{d}$ full sample**	**$\min(\hat{d})$**	**10 samples median$(\hat{d})$**	**$\max(\hat{d})$**	**# $\hat{d}$ sign. > 0 in 10 samples**
Simulation I	0.23	0.33	0.55	0.84	6
Simulation II	0.15	0.27	0.52	0.70	7
Lux&Marchesi		0.43	0.63	0.75	10

Table 6. Estimates of the long memory parameter d for squared returns (upper panel) and absolute returns (lower panel) over the full sample of 20,000 observations and over 10 subsamples of 2,000 observations each for parameter set III. The last column contains the number of significantly positive estimated long memory parameters within the ten subsamples at a significance level of 5%. Significantly positive estimates of d over the full sample are marked with an asterix (*).

Squared Returns	$\hat{d}$ full sample	10 samples min($\hat{d}$)	median($\hat{d}$)	max($\hat{d}$)	# $\hat{d}$ sign. > 0 in 10 samples
Simulation I	0.07	0.12	0.47	0.73	6
Simulation II	0.35*	0.25	0.53	0.72	7
Lux&Marchesi		0.29	0.50	0.80	10
Absolute Returns	**$\hat{d}$ full sample**	**10 samples min($\hat{d}$)**	**median($\hat{d}$)**	**max($\hat{d}$)**	**# $\hat{d}$ sign. > 0 in 10 samples**
Simulation I	0.36*	0.31	0.56	0.86	8
Simulation II	0.47*	0.41	0.62	0.81	9
Lux&Marchesi		0.26	0.64	0.81	10

Table 7. Estimates of the long memory parameter d for squared returns (upper panel) and absolute returns (lower panel) over the full sample of 20,000 observations and over 10 subsamples of 2,000 observations each for parameter set IV. The last column contains the number of significantly positive estimated long memory parameters within the ten subsamples at a significance level of 5%. Significantly positive estimates of d over the full sample are marked with an asterix (*).

Squared Returns	$\hat{d}$ full sample	10 samples min($\hat{d}$)	median($\hat{d}$)	max($\hat{d}$)	# $\hat{d}$ sign. > 0 in 10 samples
Simulation I	0.22	0.26	0.47	0.64	6
Simulation II	0.19	0.15	0.40	0.77	4
Lux&Marchesi		0.20	0.52	0.70	9
Absolute Returns	**$\hat{d}$ full sample**	**10 samples min($\hat{d}$)**	**median($\hat{d}$)**	**max($\hat{d}$)**	**# $\hat{d}$ sign. > 0 in 10 samples**
Simulation I	0.32*	0.35	0.61	0.81	8
Simulation II	0.37*	0.47	0.58	0.73	10
Lux&Marchesi		0.17	0.64	0.88	9

Table 8. Results of unit root tests as reported in Lux & Marchesi (2000: table 1) upon 40 subsamples of 500 observations for each of the four parameter sets. $\hat{\rho} < 1$ and $\hat{\rho} > 1$ stand shorthand for the number of rejections of $\rho = 1$ in favour of $\rho < 1$ in one-sided tests at 95% level, and the number of rejections of $\rho = 1$ in favour of $\rho > 1$ in two-sided tests at 95% level, respectively.

Parameters	range of $\hat{\rho}$	$\hat{\rho} < 1$	$\hat{\rho} > 1$
Parameter Set I	0.999819 – 1.000022	0	0
Parameter Set II	0.999977 – 1.000021	0	0
Parameter Set III	0.999959 – 1.000030	0	3
Parameter Set IV	0.999972 – 1.000014	0	2

the subsamples, a majority of d-estimates point at long range dependence in absolute returns for all parameter sets except I, but in squared returns only for parameter set III and simulation I of parameter set IV. Overall it appears that while volatility clustering and heavy tails of the unconditional return distribution are a robust result, the finding of long memory in volatility depends somewhat stronger on the choice of the model parameters. This contrasts with the findings presented by Lux & Marchesi (2000), who report a clear majority of significantly positive d-estimates for both squared and absolute returns in all parameter sets, except for squared returns in parameter set I.

The difference might be due to the choice of the highest Fourier frequency λ_m considered in the logperiodogram regressions (2.26) of section 2.6, which is somewhat arbitrary in a similar way as the choice of the tail region in the Hill estimator of the tail index. Beran (1994: chapter 4.6) shows that different choices of m may have considerable effects upon the values and confidence intervals of $\hat{d}$. I have chosen m as the largest integer smaller than the square root of available observations in accordance with Lux (1996a). If Lux & Marchesi (2000) have chosen a different value in their study, the results regarding the significance of $\hat{d}$ may well differ. In any case, it is evident from the simulations above that the model is capable of generating long memory in volatility for at least some parameters.

As documented in table 8, Lux & Marchesi claimed originally that their simulated price series contain a unit root. All slope parameters ρ in 160 regressions of prices p_t upon their lagged values p_{t-1} are confined to a narrow range around 1, with the only

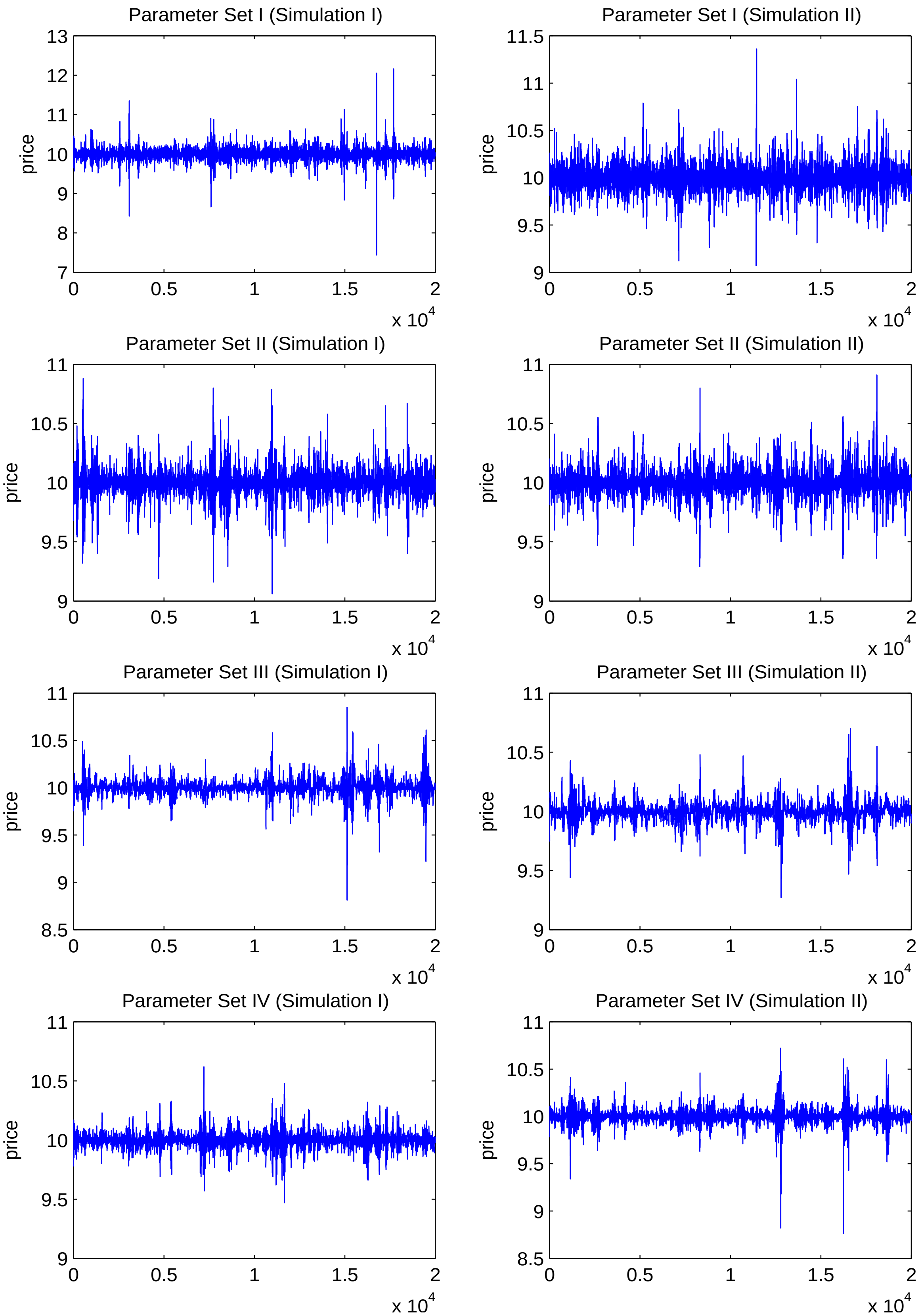

Figure 7. Price series over the full sample of 20,000 observations.

Table 9. Results of Dickey Fuller tests including a constant using the algorithm in appendix A4 upon 40 subsamples of 500 observations each. $\hat{\rho} < 1$ and $\hat{\rho} > 1$ stand shorthand for the number of rejections of $\rho = 1$ in favour of $\rho < 1$ in one-sided tests at 95% level, and the number of rejections of $\rho = 1$ in favour of $\rho > 1$ in two-sided tests at 95% level, respectively.

	Simulation I: range of $\hat{\rho}$	$\hat{\rho}<1$	$\hat{\rho}>1$	Simulation II: range of $\hat{\rho}$	$\hat{\rho}<1$	$\hat{\rho}>1$
Param. Set I	−0.37305– 0.786101	40	0	−0.04483– 0.660879	40	0
Param. Set II	0.397902 – 0.838886	40	0	0.448746 – 0.828751	40	0
Param. Set III	0.688496 – 0.095410	40	0	0.713593 – 0.949288	40	0
Param. Set IV	0.600552 – 0.871400	40	0	0.357124 – 0.848559	40	0

rejections of $H_0 : \rho = 1$ occuring in favour of explosive roots in a handful of tests for parameter sets III and IV, which they attribute to temporary instability in periods when the fraction of chartists z is close to its critical value $z_{\max}$.

Visual inspection of the simulated prices in figure 7 does not support their conjecture. All price series look clearly mean reverting around their fundamental value $p_f = 10$, instead of ever increasingly deviating from it as would be the case for integrated time series. The results of Dickey Fuller tests of the form $\Delta p_t = (\rho - 1)p_{t-1} + \text{const.} + \epsilon_t$ reported in table 9 confirm this picture. As expected, even when using subsamples of only 500 observations each as in Lux & Marchesi (2000), application of the Dickey-Fuller test using the algorithm presented in appendix A4 rejects the null of a unit root in favour of a mean reverting process with $\rho < 1$ for all of the 320 subsamples considered. The results do not change under replacement of the critical values by Fuller (1976) with the more recent ones by MacKinnon (1994).

In an attempt to spot the reason for the difference between my results and those by Lux & Marchesi, I performed also Dickey-Fuller tests of the form $\Delta p_t = (\rho - 1)p_{t-1} + \epsilon_t$ without constant using the algorithm in appendix A5, with results reported in panel a) of table 10. The estimates $\hat{\rho}$ do now indeed confine to a narrow range around $\rho = 1$ and there is no rejection of the null, neither in favour of a mean reverting process nor of an explosive root. The only way I found to additionally produce a handful significantly positive ρ estimates as in Lux & Marchesi was replacing the applicable first panel of

Table 10. Results of Dickey Fuller tests without constant using the algorithm in appendix A5 upon 40 subsamples of 500 observations each. Panel a) uses the applicable first panel of table 8.5.2 in Fuller (1976) for regressions without a constant as critical values. Panel b) applies the inapplicable second panel of the same table, as if the regression had been performed including a constant. $\hat{\rho} < 1$ and $\hat{\rho} > 1$ stand shorthand for the number of rejections of $\rho = 1$ in favour of $\rho < 1$ in one-sided tests at 95% level, and the number of rejections of $\rho = 1$ in favour of $\rho > 1$ in two-sided tests at 95% level, respectively.

a) correct table	Simulation I: range of $\hat{\rho}$	$\hat{\rho}<1$	$\hat{\rho}>1$	Simulation II: range of $\hat{\rho}$	$\hat{\rho}<1$	$\hat{\rho}>1$
Param. Set I	0.999438 – 1.000058	0	0	0.999690 – 1.000003	0	0
Param. Set II	0.999882 – 1.000026	0	0	0.999886 – 1.000022	0	0
Param. Set III	0.999867 – 1.000067	0	0	0.999956 – 1.000067	0	0
Param. Set IV	0.999895 – 1.000036	0	0	0.999901 – 1.000038	0	0
b) wrong table	**Simulation I: range of $\hat{\rho}$**	**$\hat{\rho}<1$**	**$\hat{\rho}>1$**	**Simulation II: range of $\hat{\rho}$**	**$\hat{\rho}<1$**	**$\hat{\rho}>1$**
Param. Set I	0.999438 – 1.000058	0	0	0.999690 – 1.000003	0	0
Param. Set II	0.999882 – 1.000026	0	0	0.999886 – 1.000022	0	0
Param. Set III	0.999867 – 1.000067	0	5	0.999956 – 1.000067	0	5
Param. Set IV	0.999895 – 1.000036	0	1	0.999901 – 1.000038	0	1

table 8.5.2 in Fuller (1976), containing the critical values for regressions without a constant, with the inapplicable second panel of the same table for regressions including a constant, the results of which are reported in panel b) of table 10.

While Lux & Marchesi do not state which equation and critical values they used in performing regressions of the Dickey-Fuller type, the results reported above suggest that they have performed the tests without a constant but possibly used critical values as if the regressions had been performed including a constant, unless the differences between table 8 and panel a) of table 10 are due to the different samples. The main concern here is not so much the possible use of incorrect critical values but rather the applicability of the Dickey-Fuller test without a constant, when a constant is in fact suggested by price fluctuations around the non-zero fundamental price $p_f = 10$ in figure 7. Also, if the constant could indeed be omitted, one would not expect such large differences between the estimated AR(1) coefficients in the Dickey Fuller tests including a constant of table 9 on one hand, and their values in regressions without a constant in tables 8 and 10 on the other hand. The results in table 11, which contains the parameter estimates in

Table 11. Parameter estimates in Dickey Fuller tests of the form $\Delta p_t = (\rho - 1)p_{t-1} +$ const. $+ \epsilon_t$ upon the full samples of 20,000 observations with t-statistics in parantheses. The 1% critical values are -3.43 for $\rho-1$ (one-sided), and 2.58 for the constant (two-sided), implying significantly positive constants and rejection of a unit root in all tests.

	Simulation I:		Simulation II:	
	$(\hat{\rho}-1)$	const.	$(\hat{\rho}-1)$	const.
Parameter Set I	-0.7281	7.2803	-0.6125	6.1261
	(-107.00)	(106.99)	(-93.97)	(93.96)
Parameter Set II	-0.3421	3.4203	-0.3140	3.1405
	(-64.25)	(64.25)	(-61.04)	(61.03)
Parameter Set III	-0.1435	1.4355	-0.1208	1.2075
	(-39.36)	(39.36)	(-35.89)	(35.88)
Parameter Set IV	-0.2489	2.4888	-0.3543	3.5430
	(-53.34)	(53.34)	(-65.65)	(65.65)

Dickey Fuller tests of the form $\Delta p_t = (\rho - 1)p_{t-1} +$ const. $+ \epsilon_t$ upon the full samples of 20,000 observations, demonstrates that the constant may indeed not be omitted, as it is significantly non-zero at 99% level in all tests. Furthermore, all tests produce test-statistics for $\rho - 1$ far below the 1% critical value for rejection of a unit root, confirming our conjecture from figure 7 that the price series are in fact level stationary. We can therefore not confirm the claim brought forward by Lux & Marchesi, that the model with quasi-meanvalue dynamics (4.51) would generate integrated price series under the assumption of a constant fundamental price p_f, but attribute it to a flawed application of the Dickey-Fuller test in their study.

As has been mentioned earlier in the introduction of section 4.3, this should not be regarded as a criticism of the model, since the failure of the simulations to produce integrated prices as in real financial data is only due to the unrealistic assumption of a constant fundamental value. Lux & Marchesi (1999) show that assuming p_f to follow geometric Brownian motion does indeed lead to integrated price series with otherwise similar statistical properties as presented here.

We shall now turn to traders holdings and cash, an issue which has not been investigated by Lux & Marchesi. I have several times mentioned the criticism by Farmer

& Joshi (2002) of the order based setup by Beja & Goldman (1980) which was also applied in their study,[134] as the uncoupling of orders and acquired positions may lead to unbounded inventories. The underlying reason not explained by Farmer & Joshi is the following: Orders, if filled, are derivatives of traders holdings with respect to time; or stated the other way round, traders holdings are the integral of filled orders over time. Now if orders, rather than holdings, are assumed to follow a level stationary process, such as the stationary levels of mispricing and the number of chartists in equation (4.50), then integrating over these orders in order to obtain traders inventories will generate integrated and therefore unbounded holdings, unless the stationary series which the trading decisions were based upon were already over-differenced. This will now be exemplified using the faster executing code of simulations II presented in appendix A1 with binomial random variate generation using the BINV algorithm.

Since individual traders may change their strategy at any time, we shall look at aggregate holdings and cash of the entire fundamentalist and chartist subpopulations, rather than those of single traders. As the model allows for unlimited buying and short selling of stocks, each group is initialized to hold neither stocks nor cash (lines 143–146 in the code presented in appendix A1). Once the excess demand *ED* in equation (4.50) is determined, chartists inventories are increased by $ED_c = (n_+ - n_-)t_c$ (line 209) and fundamentalist inventories by $ED_f = n_f t_f (p_f - p)$ (line 210). The corresponding amounts of cash, $p \cdot ED_{c/f}$, are subtracted from their wealth in lines 217–218 following the price adjustment in lines 212–214. The aggregate wealth of the two trader subpopulations equal their aggregate inventories evaluated at market price plus their cash (lines 254–255). Since the market maker has to supply the shares demanded by the fundamentalist and chartist trader populations starting with zero inventories and cash, her holdings and cash equal the traders aggregate inventories and cash, however with opposite signs (lines 277–278).

Consider first the aggregate holdings of chartists and fundamentalists over 20,000 observations in figure 8. During that time, traders accumulate inventories of the same order of magnitude as the number of observations, with close to symmetric portfolio holdings for chartists and fundamentalists. With the possible exception of parame-

[134] see equation (4.50)

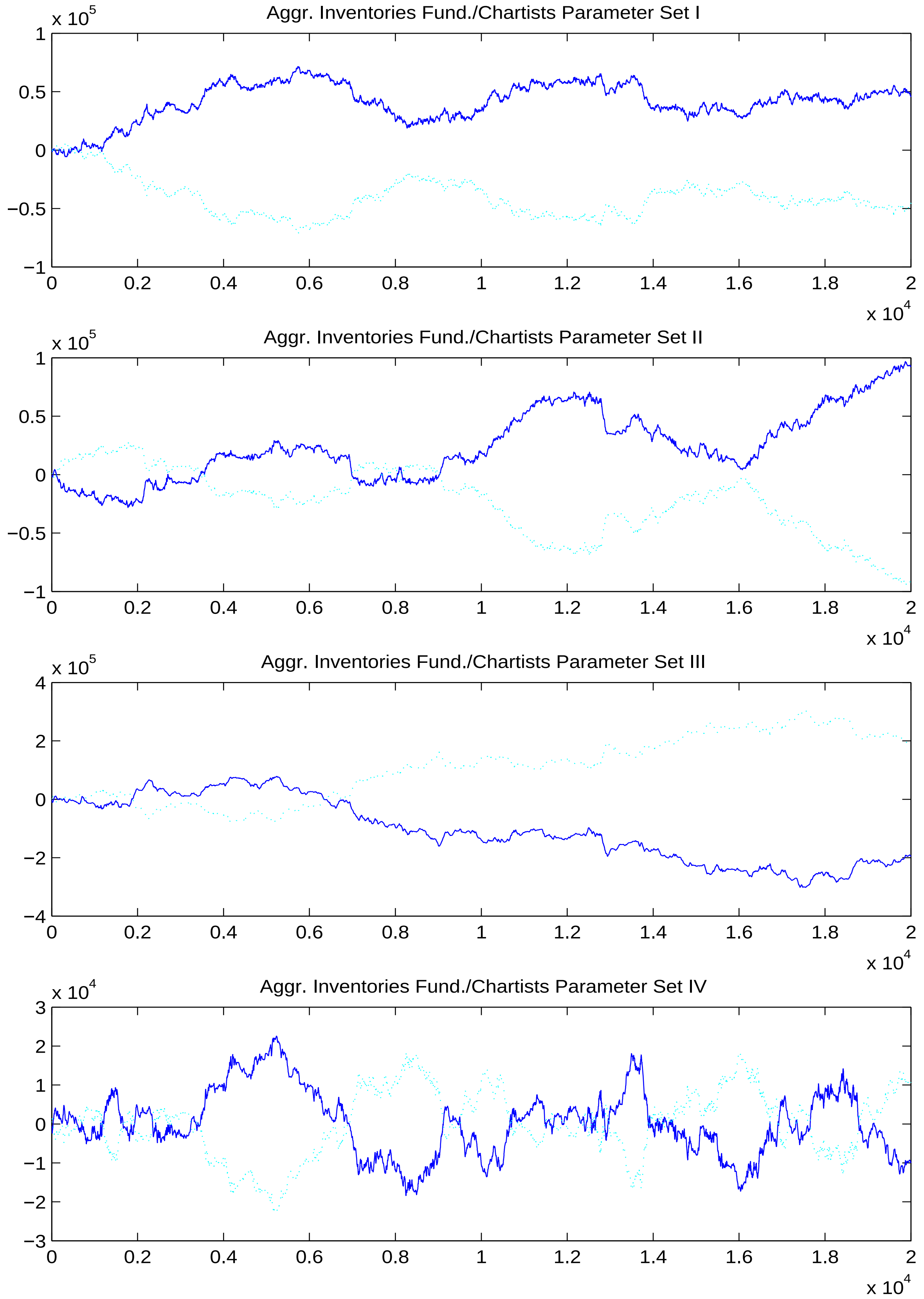

Figure 8. Aggregate holdings of the chartist (dark solid line) and the fundamentalist subgroup (light dotted line) over 20,000 observations.

Table 12. Results of augmented Dickey Fuller tests with automatic lag length selection based upon SIC on aggregate chartist and fundamentalist holdings over the full sample of 20,000 observations. p-values denote the probability of falsely rejecting $H_0: \rho \geq 1$ using the one-sided critical values by MacKinnon (1996).

Parameters	$\hat{\rho}$ Chartists	$\hat{\rho}$ Fundam.	p-value Chartists	p-value Fundam.
Parameter Set I	0.999841	0.999841	0.3585	0.3552
Parameter Set II	0.999946	0.999847	0.8365	0.8413
Parameter Set III	0.999978	0.999971	0.5293	0.5362
Parameter Set IV	0.999838	0.999837	0.2034	0.1961

Table 13. Results of augmented Dickey Fuller tests with automatic lag length selection based upon SIC on market maker holdings over the full sample of 20,000 observations. p-values denote the probability of falsely rejecting a unit root or a non zero trend.

Parameters	$\hat{\rho}$	(p-value)	trend	(p-value)
Parameter Set I	0.99859	(0.228)	-0.000022	(0.1957)
Parameter Set II	0.99833	(0.0411)	-0.000187	(0.0009)
Parameter Set III	0.998992	(0.0661)	-0.000191	(0.0011)
Parameter Set IV	0.999719	(0.856)	-0.00002	(0.2616)

ter set IV, inventories appear to follow rather random walk like than mean reverting processes. This view is confirmed by insepecting the results of augmented Dickey-Fuller tests upon traders inventories in table 12. None of the tests led to a rejection of a unit root in inventories even in these very large samples. The only sample which comes somewhat close to a rejection of a unit root is parameter set IV with p-values around 0.2 for both the holdings of chartists and fundamentalists. Also, the holdings do not visually appear unbounded in this sample. One might therefore argue, that even longer data sets would finally lead to a rejection of a unit root at least for parameter set IV. However, the closeness to level-stationary holdings in this parameter set could well be due to the periodicity of the random number generator. The generation of 20,000 data points required 8×10^7 calls of the random number generator. This is only by a factor of 25 below the largest positive value representable by signed 32-bit integers of 2×10^9, which is an upper limit for the period of any random number generator of the form $X_{n+1} = f(X_n)$ on 32-bit computers[135].

[135] See for example Robert & Casella (2004).

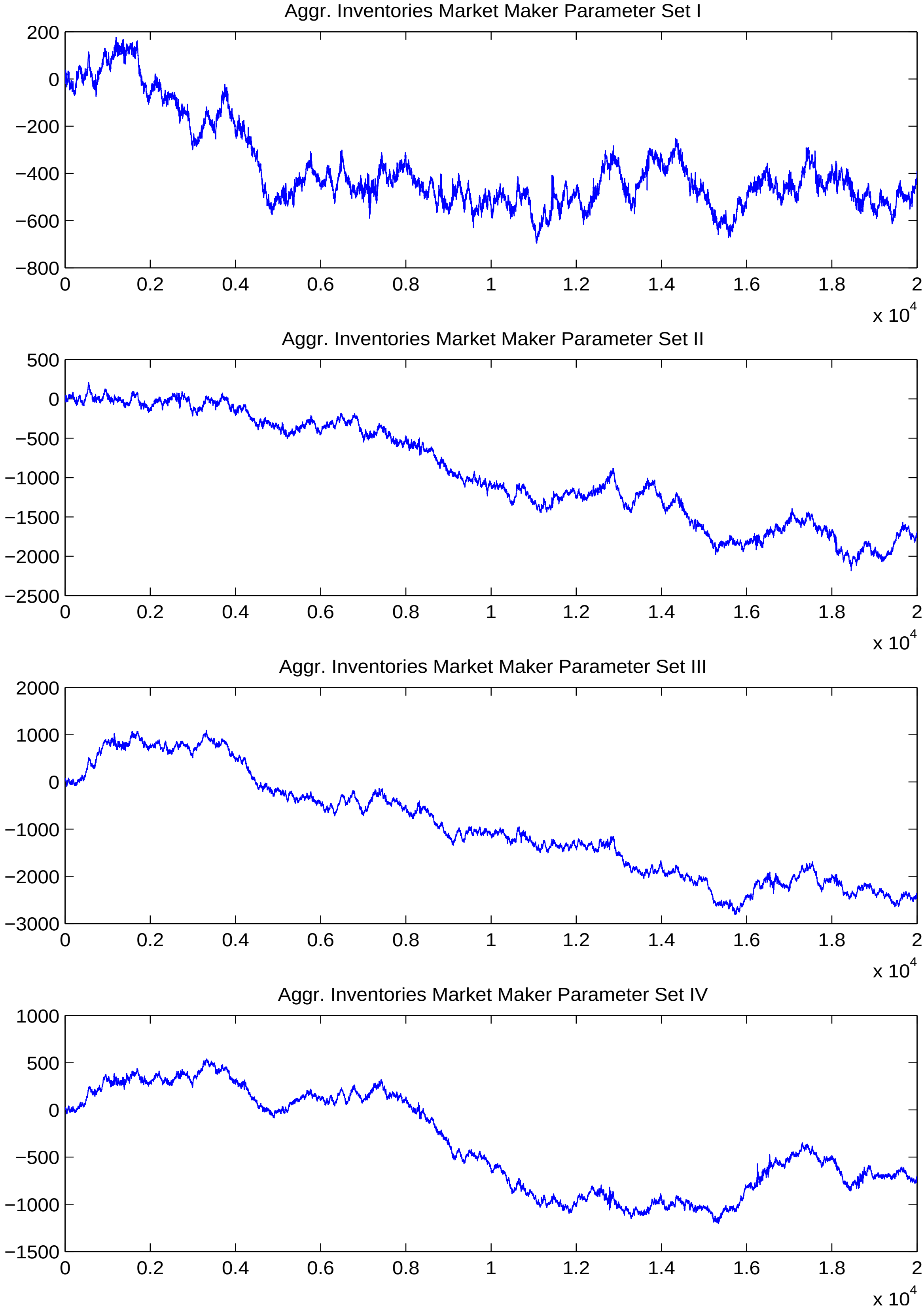

Figure 9. Aggregate inventories of the market maker over 20,000 observations.

Consider next the market maker inventories in figure 9. The results of the augmented Dickey-Fuller tests together with the deterministic trend parameters are depicted in table 13. None of the series are bounded, as they contain either a unit root (Parameter sets I, IV) or have a significant deterministic trend (Parameter sets II, III), consistent with the before mentioned possibility of building up unbounded inventories.

Consider finally the wealth dynamics for the different kind of traders. Figure 10 plots the aggregate wealth of the chartist and fundamentalist subpopulations over time. It is immediately evident that chartists loose their money to fundamentalists for all parameter sets considered. While one may be tempted to conclude that this will cause chartists to die out in course of time, this need not necessarily be so for at least three reasons:

1. Since traders are constantly changing between a chartist and a fundamentalist strategy, traders may well recover losses experienced while using a chartist strategy from the profits made when trading as fundamentalists.

2. It is reasonable to assume that fundamentalism is costlier then chartism in the sense that figuring out the true value of an asset requires more resources than just following a trend. These costs might just offset the profits fundamentalists make relative to the losses of chartists.

3. If the costs of market entry are lower for chartists than for fundamentalists, it is reasonable to assume that bankrupt chartists are replaced by new chartists entering the market.

The wealth dynamics of the marketmaker is depicted in figure 11. In all four cases market makers incur losses at close to constant rates, which appear harder to justify than those of the chartists because market makers don't change strategy. However, the market maker could well charge a fee from his trading partners in order to repair his losses, for example in form of a bid-ask spread as is common practice in financial markets.

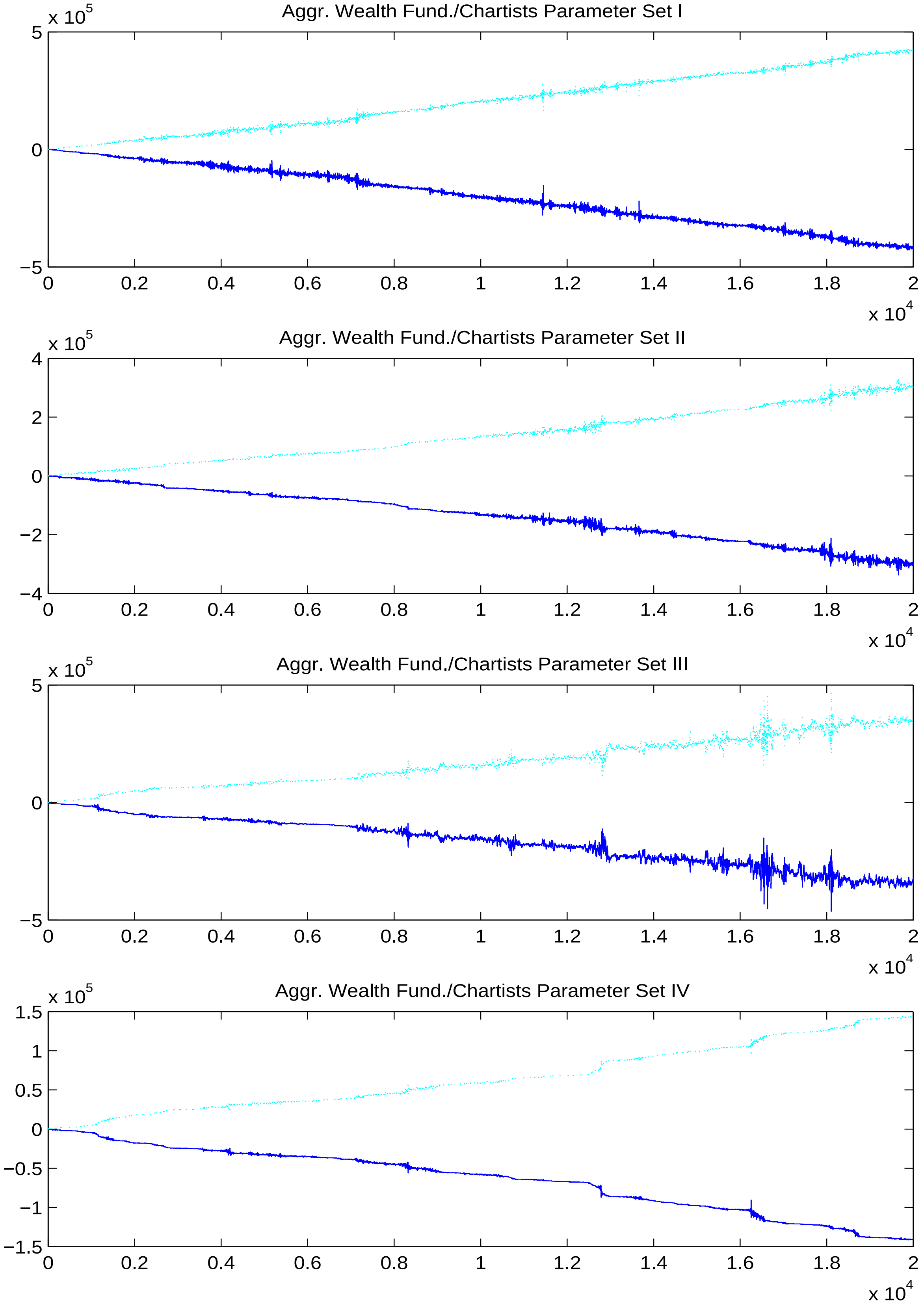

Figure 10. Aggregate wealth of the chartist (dark solid line) and the fundamentalist subgroup (light dotted line) over 20,000 observations.

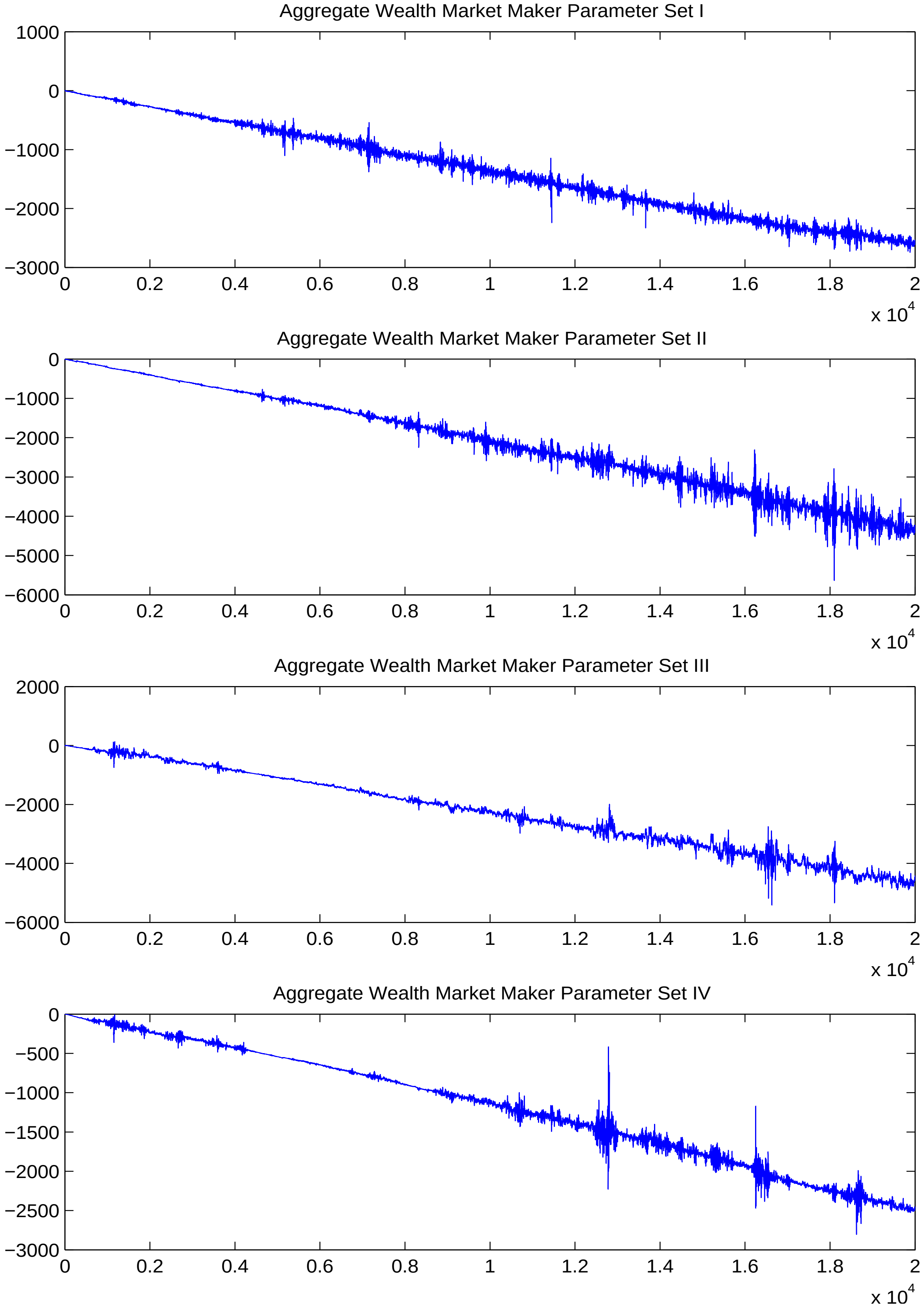

Figure 11. Aggregate wealth of the market maker over 20,000 observations.

Overall, we can at least not reject the concerns brought forward by Farmer and Joshi, that order based strategies violate fundamental risk management constraints by their implicit tendency to build up infinite holdings over time. In the model by Lux & Marchesi (2000) the problems of order based trading became particularly evident for the inventories of the market maker. This may be not so surprising, given that the market makers inventories provide the loophole for the modeller to replace equilibrium with disequilibrium trading.

Obviously, it would require considerable additional effort to include market makers wealth and positions into a consistent model of the price discovery process. One may also ask whether the disequilibrium trading provided by the market maker is really such an important feature of financial markets to model for return periods of a full trading day and above, as was the intention in Lux' model. In markets without 24 hours trading such as stock markets closing prices must be quite close to equilibrium prices, since otherwise market participants wouldn't be prepared to sleep with them until next morning.

I shall therefore drop the marketmaker in a simplified version of Lux' model with position-based trading in the next chapter. Because the model is position based, it is easy to generalize to multiple assets, avoiding the inconsistencies of order based trading discussed in the introduction and demonstrated in this section. In order to add further realism to the model, I will also include a riskless bond (cash), and separate the security selection decision between different stocks from the asset allocation decision between equity and bonds.

5 Asset Allocation and Position Based Trading

5.1 The Model

In this section we shall develope a model for the price discovery process of multiple assets in position based trading without a market maker. Consider for that purpose an investment community of N portfolio managers or traders. They hold individually only one of three assets, either one of two risky stock issues or a bond issue in infinite supply (cash). The logarithmic trading prices of the stocks are denoted by p_1 and p_2, and the logarithm of their fundamental values by p_{f1} and p_{f2}. Portfolio managers holding stocks may choose one of two investment strategies, fundamentalist or chartist. Fundamentalists hold long (short) positions in a stock because its trading price is below (above) its fundamental price, to which they expect the trading price to converge in the long run. Chartists wish to hold a stock because most market participants already own it (herding). This simplifies Lux' original setup by not explicitly including the trend of the stock price itself as a motive for holding stocks, and is done here in order to keep the mathematical formulation of the multivariate setup concise. Herding, rather than riding a price trend, was also the numerically dominant trading motive for chartists in Lux' parameter sets.[136]

Denote the number of chartists invested in stock 1 or 2 with n_{c1} resp. n_{c2} and the number of fundamentalists invested in stock 1 or 2 with n_{f1} resp. n_{f2}. Each chartist wishes to hold t_c issues of her favourite stock, whereas the desired holdings of fundamentalists are proportional to the mispricing of the stock they wish to hold. Denoting fundamentalists target holdings per unit mispricing with t_f, the aggregate target holding in either stock is

$$E_i = n_{ci}t_c + n_{fi}t_f(p_{fi} - p_i), \qquad i = 1, 2, \tag{5.1}$$

where the first and second term denote aggregate target exposure in stock i by chartists and fundamentalists, respectively. This equation may be seen as a multivariate generalization of the net excess demand $ED = ED_c + ED_f$ in equation (4.50) of the model

[136] The contributions of $\alpha_1 x$ to U_1 in equation (4.46), page 85, were about one order of magnitude larger than those of $\alpha_2 \dot{p}/v_1$ in the simulations of section 4.3.2.

by Lux (1998). The key difference is that Lux follows the order-based literature in using this expression to describe stocks to *trade* rather than target positions in stocks to *hold*, as is the case here. We do therefore expect traders holdings—unlike those in the simulations of Lux & Marchesi (2000)—to remain bounded due to level stationarity of the number of chartists n_{ci} and the mispricing $p_i - p_{fi}$. This claim will be verified in section 5.2.

I assume the target holding parameters t_c and t_f and the fundamental prices p_{f1} and p_{f2} to be constant over the time period considered. Trading demand for the stocks is generated by changes in desired aggregate holdings due to changes in mispricing or the composition of traders

$$ED_i = \frac{d}{dt}E_i = \dot{n_{ci}}t_c + \dot{n_{fi}}t_f(p_{fi} - p_i) - n_{fi}t_f\dot{p_i}, \qquad i = 1, 2. \tag{5.2}$$

Market clearing ($ED_i = 0$) yields for the logarithmic trading prices of the stocks

$$\dot{p_i} = \frac{1}{n_{fi}}\left(\dot{n_{ci}}\frac{t_c}{t_f} + \dot{n_{fi}}(p_{fi} - p_i)\right), \qquad i = 1, 2. \tag{5.3}$$

We see from equation (5.3) that fast changes in the composition of traders and large mispricings speed up price changes, whereas large fundamentalist populations slow them down. On the chartist side, the speed of price adjustment depends on the target exposures of chartists relative to fundamentalists. Large chartist exposures speed up price changes whereas large fundamentalist exposures have the opposite effect. Overall, we recover the recurrent theme from the interacting agent literature, that fundamentalists have a stabilizing effect and that noise traders have a destabilizing effect upon prices, without having made any specific assumptions yet about how to model changes in the traders populations.

Another important conclusion from equations (5.1) to (5.3) is that our trading process conserves the number of shares traded, a feature not necessarily present in order based models including a market maker, as was demonstrated in section 4.3.2. This may be seen as follows: Because we assume market clearing, the aggregate target holdings E_1 and E_2 in equation (5.1) must equal the number of shares issued by companies 1 and 2.

The condition $d\,E_i/dt = 0$ for market clearing implies then, that the respective number of shares remains constant through time.

Let us now turn to the population dynamics of the different trader types. Like Lux & Marchesi, we follow the synergetics literature in modeling interactions between members of the investment community in terms of Markov chains. That is, for each trader we postulate a transition probability to change her state of behaviour, or equivalently to move to another subpopulation, which depends only upon the investment communities current state, as described by the respective numbers of different trader types. Suppose there are L subpopulations (trader types) $n_1, \ldots, n_i, \ldots, n_j, \ldots, n_L$ and denote the transition probability to move from subpopulation i to subpopulation j as p_{ij}. The evolution of expected population sizes through time is the described by the quasi-meanvalue equations[137]

$$\dot{n}_i = \sum_{j=1}^{L}(n_j p_{ji} - n_i p_{ij}), \qquad i = 1, \ldots, L. \tag{5.4}$$

Intuitively, they state that the expected change in population size per time unit $\dot{n}_i$ consists of expected population inflows from all other states $\sum n_j p_{ji}$ minus all expected population outflows into other states $\sum n_i p_{ij}$. In our case we have $L = 5$ subpopulations: two chartist populations of size n_{c1} and n_{c2}, two fundamentalist populations of size n_{f1} and n_{f2}, and one bondholder population of size

$$n_B := N - n_E, \quad \text{where} \quad n_E := n_{c1} + n_{c2} + n_{f1} + n_{f2} \tag{5.5}$$

denotes the number of equity investors. Our task is now to specify the transition probabilities p_{ij} according to which traders change from one subgroup to another. As in Lux, it is assumed that traders change their strategy according to the perceived profits of the other strategies compared to their own. The perceived profits or utilities F_i of fundamentalists holding a position in stock i are modeled as

$$F_i = s|p_{fi} - p_i|, \qquad i = 1, 2, \tag{5.6}$$

[137] see appendix A7 for a derivation. The triangular brackets $< \, . \, >_t$ to indicate the expectation operator applied to the process at time t have been dropped for notational convenience.

where s is a discount factor, since reversals to the fundamental price are expected to occur only in the future. The fundamentalist utility F_i is thus proportional to the logarithmic mispricing in stock i, similar to the model by Lux (1998).

The utility of chartists is assumed as

$$C_i = \frac{n_{ci} + n_{fi} - n_{cj} - n_{fj}}{N}, \quad i, j = 1, 2, \quad i \neq j. \tag{5.7}$$

This generalizes the opinion index x—defined in (4.44) as an input for herding in Lux' model—to multiple assets, as it describes the scaled difference between equity investors in stock i and j. The more traders there are invested in stock i relative to stock j, the more attractive stock i becomes relative to stock j (herding), and the higher the chartist utility C_i in stock i will be relative to the chartist utility C_j in stock j.

I follow Lux in assuming that the relative change in probability to switch from one strategy to another is proportional to the difference between the utilities of the respective strategies,[138] i.e.

$$dp_{ij}/p_{ij} = \alpha\, d(U_j - U_i) \quad \text{with} \quad U_i, U_j \in \{C_1, C_2, F_1, F_2\}, \tag{5.8}$$

where α measures the strengh of attraction which apparently more profitable strategies exert upon the trader. Inserting the utilities (5.6) and (5.7) into (5.8) yields for the transition probabilities between the trader types

$$\begin{aligned} p_{cicj} &= ve^{\alpha(C_j - C_i)}, \quad p_{fifj} = ve^{\alpha(F_j - F_i)}, \qquad i, j = 1, 2, \; i \neq j \\ p_{cifj} &= ve^{\alpha(F_j - C_i)}, \quad p_{ficj} = ve^{\alpha(C_j - F_i)}, \qquad i, j = 1, 2, \end{aligned} \tag{5.9}$$

where p_{cicj} and p_{fifj} denote transitions from stock i to stock j within the chartist and the fundamentalist subgroup respectively, and p_{cifj} and p_{ficj} denote transitions from chartists to fundamentalists and vice versa. The speed of adjustment parameter v measures the frequency at which equity investors reconsider their investment strategy and depends therefore upon the time unit chosen in the description of the dynamic process.

Consider next the transitions between bond and equity investors as illustrated in figure 12. We assume that asset allocation and security selection are performed by separate

[138] see property *ii)* in the discussion of (4.37) on page 82.

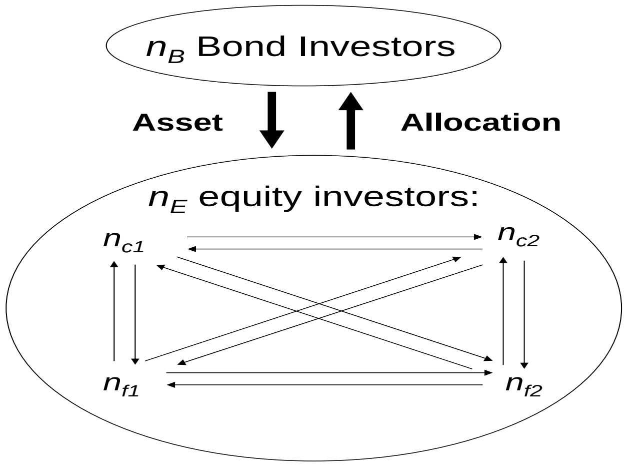

Figure 12. Security selection and asset allocation are modeled as separate decision processes. The sponsor decides how many traders to put onto the fixed income as opposed to the equity side (asset allocation). Portfolio managers decide about the stock to invest in and the trading strategy to use (security selection).

entities, as is common practice in financial institutions[139]. That is, the individual trader or portfolio manager has no freedom to decide whether to invest in stocks or bonds, but chooses only specific securities within his asset class. This corresponds to portfolio managers in the majority of financial institutions, managing either an equity or a fixed income portfolio.

The decision how to split up traders between the equity and the fixed income side is done by a separate entity, which we shall call the asset allocator or sponsor. He or she is often an external client with little market information who wishes to delegate the investment management to professionals, whereas the before mentioned security selection is usually done by professional portfolio managers in house. Even when both asset allocation and security selection decisions are made in the same financial institution, the former are generally done by upper hierarchy levels. These have usually more duties than just making asset allocation decisions, which may prevent them from

[139] An in-depth treatment of the institutional investment process is provided by Davis and Steil Davis & Steil (2001).

processing valuation relevant information as efficiently as their portfolio managers at security selection level do. The utility of equity investments for the sponsor is therefore modeled in the same spirit as that of the chartists as

$$E = \frac{n_E - n_B}{N}. \tag{5.10}$$

That is, the more equity (bond) investors there are already in the market, the more attractive equity (fixed income) investment becomes for the sponsor.

For the sake of simplicity, the perfectly elastically supplied bond (cash) is assumed to pay no interest, such that its utility is zero. The resulting transition rates between equities and bonds read then in analogy to (5.9)

$$p_{BE} = v_B e^{\alpha_B (n_E - n_B)/N} \qquad \text{and} \qquad p_{EB} = v_B e^{-\alpha_B (n_E - n_B)/N}, \tag{5.11}$$

where α_B is the strength of infection parameter between equity and bonds and v_B denotes the frequency at which asset allocators reconsider their strategy.

In the next step we need to specify, how the transitions between equity and bonds on asset allocation level translate into transition probabilities between the individual stock investors and the bondholders. Keeping in mind that institutional investment practice demands asset allocation and security selection to be modeled as separate processes, I shall assume here that the asset allocation decision leaves the internal composition of stock investors unchanged. That is, the transition rates from each individual stock investor to bondholders equal just the transition rates between equity and bonds

$$p_{ciB} = p_{fiB} = p_{EB}, \qquad i = 1, 2, \tag{5.12}$$

whereas transitions from the bondholders to the equity investors must be weighted by the relative frequency of the relevant stock investor type

$$p_{Bci} = \frac{n_{ci}}{n_E} p_{BE}, \qquad p_{Bfi} = \frac{n_{fi}}{n_E} p_{BE}, \qquad i = 1, 2. \tag{5.13}$$

These may then be inserted into the quasi-meanvalue equations (5.4) in order to obtain for the population dynamics:

$$
\begin{aligned}
\dot{n_{c1}} = v_B\ n_{c1} & \left(\frac{n_B}{n_E} e^{\alpha_B (n_E - n_B)/N} - e^{-\alpha_B (n_E - n_B)/N} \right) \\
+ v \Big[& \ n_{c2} e^{\alpha(C_1 - C_2)} - n_{c1} e^{\alpha(C_2 - C_1)} \\
& + n_{f1} e^{\alpha(C_1 - F_1)} - n_{c1} e^{\alpha(F_1 - C_1)} \\
& + n_{f2} e^{\alpha(C_1 - F_2)} - n_{c1} e^{\alpha(F_2 - C_1)} \Big]
\end{aligned}
\tag{5.14a}
$$

$$
\begin{aligned}
\dot{n_{c2}} = v_B\ n_{c2} & \left(\frac{n_B}{n_E} e^{\alpha_B (n_E - n_B)/N} - e^{-\alpha_B (n_E - n_B)/N} \right) \\
+ v \Big[& \ n_{c1} e^{\alpha(C_2 - C_1)} - n_{c2} e^{\alpha(C_1 - C_2)} \\
& + n_{f1} e^{\alpha(C_2 - F_1)} - n_{c2} e^{\alpha(F_1 - C_2)} \\
& + n_{f2} e^{\alpha(C_2 - F_2)} - n_{c2} e^{\alpha(F_2 - C_2)} \Big]
\end{aligned}
\tag{5.14b}
$$

$$
\begin{aligned}
\dot{n_{f1}} = v_B\ n_{f1} & \left(\frac{n_B}{n_E} e^{\alpha_B (n_E - n_B)/N} - e^{-\alpha_B (n_E - n_B)/N} \right) \\
+ v \Big[& \ n_{c1} e^{\alpha(F_1 - C_1)} - n_{f1} e^{\alpha(C_1 - F_1)} \\
& + n_{c2} e^{\alpha(F_1 - C_2)} - n_{f1} e^{\alpha(C_2 - F_1)} \\
& + n_{f2} e^{\alpha(F_1 - F_2)} - n_{f1} e^{\alpha(F_2 - F_1)} \Big]
\end{aligned}
\tag{5.14c}
$$

$$
\begin{aligned}
\dot{n_{f2}} = v_B\ n_{f2} & \left(\frac{n_B}{n_E} e^{\alpha_B (n_E - n_B)/N} - e^{-\alpha_B (n_E - n_B)/N} \right) \\
+ v \Big[& \ n_{c1} e^{\alpha(F_2 - C_1)} - n_{f2} e^{\alpha(C_1 - F_2)} \\
& + n_{c2} e^{\alpha(F_2 - C_2)} - n_{f2} e^{\alpha(C_2 - F_2)} \\
& + n_{f1} e^{\alpha(F_2 - F_1)} - n_{f2} e^{\alpha(F_1 - F_2)} \Big]
\end{aligned}
\tag{5.14d}
$$

Combining the time development of the asset prices (5.3) with the population dynamics (5.14) one obtains a self-contained system of highly non-linear differential equations with state variables p_1, p_2, n_{c1}, n_{c2}, n_{f1} and n_{f2}. It turns out that this system has a "fundamental"equilibrium, in which the trading prices of both assets equal their respective fundamental values with balanced disposition among traders as detailed below.

Proposition 1. *Existence of a fundamental equilibrium.*
The market with separate asset allocation has a fundamental equilibrium at

$$n_B = n_E = N/2, \qquad n_{c1} = n_{c2} = n_{f1} = n_{f2} = n_E/4 = N/8.$$

Proof. See appendix A8. □

Intuitively, the equilibrium conditions follow quite naturally from the structure of the quasi-meanvalue equations (5.4) as follows. Consider first the subdynamics of the equity investor populations. At fundamental equilibrium both fundamentalist utilities equal zero, because all trading prices equal their fundamental values. Also the chartist utilities equal zero when there are equally many equity investors in stock 1 and 2. All transition probabilities between equity investments in the quasi-meanvalue equations equal then v, such that (5.4) simplifies for the subdynamics between equity investors to

$$\dot{n}_i = v \cdot \sum_{j=1}^{4} (n_j - n_i), \qquad n_i, n_j = n_{c1}, n_{c2}, n_{f1}, n_{f2}. \tag{5.15}$$

It is then immediately clear from (5.15) that zero expected changes for all trader populations imply that there are equally many investors in each of the equity strategy subpopulations. The same argument applies for the asset allocation subdynamics, thereby implying equally many equity and bond investors.

Employment of absolute values in the fundamentalist utilities (5.6) implies that the system of differential equations (5.3) and (5.14) contains four subregimes ($p_1 > p_{f1}$, $p_2 > p_{f2}$), ($p_1 < p_{f1}$, $p_2 < p_{f2}$), ($p_1 > p_{f1}$, $p_2 < p_{f2}$), and ($p_1 < p_{f1}$, $p_2 > p_{f2}$). Necessary conditions for simultaneous stability of the fundamental equilibrium with respect to the regime-specific dynamics are detailed in proposition 2 below. Note however, that stability with respect to the regime-specific dynamics is in general neither a sufficient nor necessary condition for stability of the overall dynamics. E.g. Honkapohja & Ito (1983) provide several examples demonstrating that stable regimes may very well be patched into an unstable system when a trajectory crosses boundaries at a series of points which become further and further displaced from the equilibrium, or a solution path slides along a boundary in a direction divergent from the equilibrium point. Proposition 2 serves therefore only as a general guideline, which factors may have an impact upon local stability of the fundamental equilibrium within the overall dynamics.

Proposition 2. *Local stability with respect to regime-specific dynamics.*
The following are necessary conditions for simultaneous local stability of the fundamental equilibrium with respect to all four subregimes:

1. $$\alpha_B \leq 1,$$

2.
$$\begin{aligned} \alpha(1+\sqrt{1+16ls}) &\leq 4 \quad \textit{for} \quad ls \leq 3/2, \\ \alpha ls &\leq 1 \quad\quad\;\; \textit{for} \quad ls \geq 3/2, \end{aligned}$$

$$\textit{with} \quad l := t_c/t_f$$

Proof. See appendix A9. □

The above conditions for local stability of the fundamental equilibrium with respect to the regime-specific dynamics conform with intuition. Large strength of attraction parameters imply that small deviations from equilibrium trigger fast changes in the trader populations, leading to fast price changes as well. Large holdings of chartists relative to fundamentalists speed up price changes as was already mentioned in the discussion of (5.3). Large discount factors have a similar effect in speeding up population changes by their inclusion into the transition rates between equity investors (5.9) through the fundamentalist utilities (5.6).

5.2 Simulation Study

We shall in the following simulate the artificial market defined by equations (5.3) and (5.14) along the same lines as in Lux & Marchesi (2000). That is, we consider an ensemble of $N = 500$ traders with asynchronous updating of strategies approximated by finite time increments of size $\Delta t = 0.002$ in the domain of attraction of the fundamental equilibrium. In order to initialize the simulations both trading prices are set to their fundamental value, while the numbers of chartists and fundamentalists in each stock are set to 62 and 63 respectively, close to their fundamental equilibrium value of $500/8 =$

Table 14. Parameter set used in the simulations.

p_{f1}	p_{f2}	v	v_B	α	α_B	$l = t_c/t_f$	s
0	0	0.001	0.04	0.1	0.4	0.5	0.8

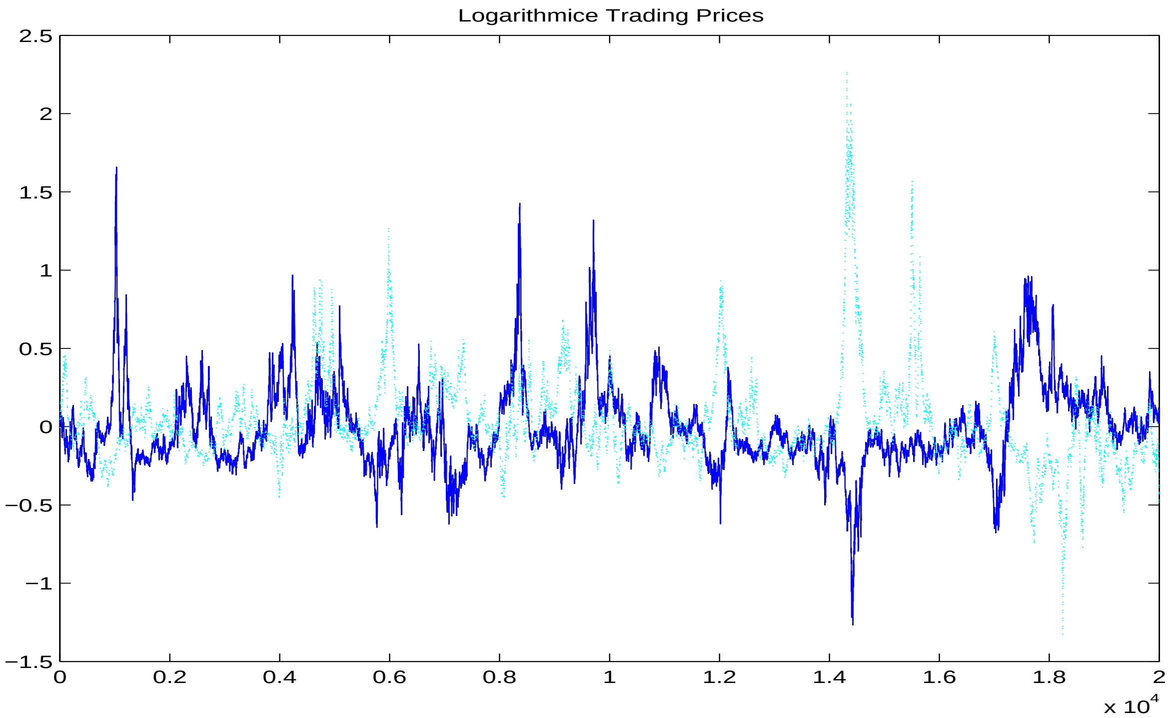

Figure 13. Logarithmic trading prices p_1 (dark solid line) and p_2 (light dotted line) for the two risky stocks over 20,000 observations. The logarithmic intrinsic values of both stocks remain constant at $p_{f1} = p_{f2} = 0$.

62.5 identified in proposition 1. Similar as in Lux, the parameter set depicted in table 14 has been chosen using the criterion that the bandwidth for returns over unit time steps should roughly conform to what one usually observes for daily data in financial markets.

Consider first the plot of logarithmic trading prices in figure 13, where the logarithmic fundamental prices of both stocks were set to zero. Obviously the model is capable of generating both severe crashes and long lasting bubbles. At its most extreme observation, stock 2 trades at almost ten times its intrinsic value. Substantial deviations between fundamental and trading price may occur for several hundred observations in a row, corresponding to time spans of a year and above in real markets. As such, the simulated price series look certainly more realistic than those of the model by Lux & Marchesi (2000) presented in figure 7, where the crossings with zero mispricing occur so fast that they cannot even be identified in this scale. However, also the price series of our new model strongly reject the null hypothesis of a unit root as is demonstrated

Table 15. Probability values of falsely rejecting $H_0\colon\ \rho \geq 1$ (and const.$= 0$) in Dickey Fuller tests of the form $\Delta p_t = (\rho - 1)p_{t-1} + (\text{const.}) + \epsilon_t$ over the full sample of 20,000 observations.

p-values:	$\Delta p_t = (\rho - 1)p_{t-1} + \text{const.} + \epsilon_t$: ρ	const.	$\Delta p_t = (\rho - 1)p_{t-1} + \epsilon_t$: ρ
p_1	$1.95 \cdot 10^{-13}$	0.7709	$2.03 \cdot 10^{-13}$
p_2	$2.63 \cdot 10^{-11}$	0.5691	$3.08 \cdot 10^{-11}$

in table 15, no matter whether an (insignificant) constant is included into the Dickey-Fuller regressions or not. In that respect our price series look still as unsatisfactory as those of Lux & Marchesi (2000), but with the hindsight of the simulation study by Lux & Marchesi (1999) it appears likely that the failure of the simulations to produce integrated prices is again just due to the simplifying assumption of constant fundamental values. Therefore, as in Lux & Marchesi (2000), figure 13 should be mainly regarded as a visualization of the behaviourally explained difference between trading prices and fundamental values rather than trading prices as such.

The two upper panels of figure 14 contain the logreturns of the two stocks calculated as the difference between the simulated the logarithmic trading prices p_1 and p_2 over unit time steps as

$$r_{i,t} = p_{i,t} - p_{i,t-1}, \qquad i = 1, 2. \tag{5.16}$$

The third panel contains the logreturn of the equal weighted index calculated as

$$r_{EW,t} = \ln\left(\frac{1}{2}\exp(r_{1,t}) + \frac{1}{2}\exp(r_{2,t})\right). \tag{5.17}$$

Assuming a symmetric setup with equally many stocks issued by both companies, the returns of a capitalization weighted index may be calculated as

$$r_{CW,t} = \ln\left(\frac{\exp(p_{1,t-1})}{\exp(p_{1,t-1}) + \exp(p_{2,t-1})}\exp(r_{1,t}) + \frac{\exp(p_{2,t-1})}{\exp(p_{1,t-1}) + \exp(p_{2,t-1})}\exp(r_{2,t})\right), \tag{5.18}$$

which are plotted in the last panel of figure 14. All time series are clearly heteroscedastic with similar intermittent outbreaks of volatility as in figures 1 and 2 of section 4.3.2, and discussed as stylized facts of real financial returns in section 2.5.

Table 16 contains summary statistics for the above mentioned return series. All time

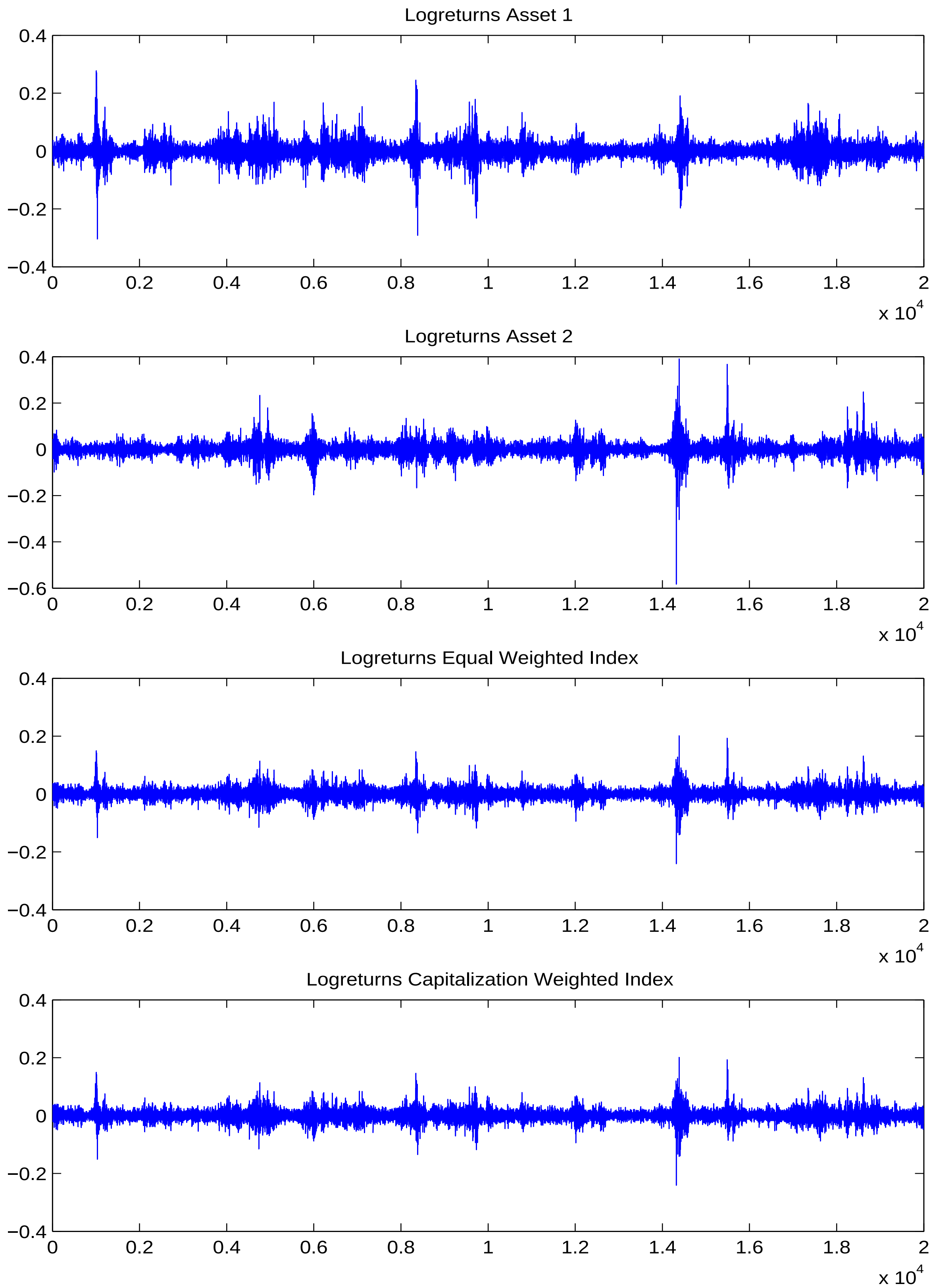

Figure 14. Logreturns of the two stocks, the equal weighted index, and the capitalization weighted index over 20,000 observations.

Table 16. Summary statistics for the simulated logreturns of the two stocks r_1 and r_2, the equal weighted index r_{EW}, and the capitalization weigthed index r_{CW}.

Asset	Avg. raw Return$\times 10^{-3}$	Avg. absolute Return	Return Variance$\times 10^{-3}$	Return Skewness	Return Kurtosis
Stock 1	0.0040	0.0180	0.7071	0.0773	12.07
Stock 2	-0.0231	0.0179	0.7251	-0.1424	25.88
Index (EW)	0.1728	0.0133	0.3514	0.1331	10.47
Index (CW)	-0.0077	0.0144	0.4886	-0.3443	37.58

series are close to symmetric, with average daily absolute returns in the range of 1 to 2 percent, and an annual volatility in the range between 30 and 40 percent. Because the individual stock returns are cross-sectionally close to uncorrelated ($\rho = -0.0195$), the variance of the equal weighted index is about half the variance of the individual stock returns. All time series are heavily leptokurtic with double digit coefficients of kurtosis. The variance and kurtosis of the capitalization weighted index are somewhat higher than those of the equal weighted index due to their higher weight on the stock with the larger mispricing and therefore higher probability of large returns.

Empirically observed stock returns are in general positively correlated, which is not replicated by the simulations presented here under the simplifying assumption of equal constant fundamental values for both stocks. If we had assumed their intrinsic values to follow correlated unit root processes, uncorrelatedness in the current setup would presumably have translated into identical correlations of fundamental and trading returns. The model may therefore very well be consistent with the positively correlated returns observed in real equity markets, as far as fundamental values are positively correlated. The positive cross-sectional correlation between stocks would just drop out from the list of behaviourally explained stylized facts and instead be attributed to economic facts such as exposure to similiar risk factors etc. In any case, I am not aware of any order-based study, which would have reported cross-sectionally uncorrelated or even positively correlated return series, as the findings from empirically oberved equity returns would require.

Figures 15 and 16 demonstrate that the outbreaks of volatility in trading returns are related to the number of stocks owned by chartists in a similar way as in the simulation

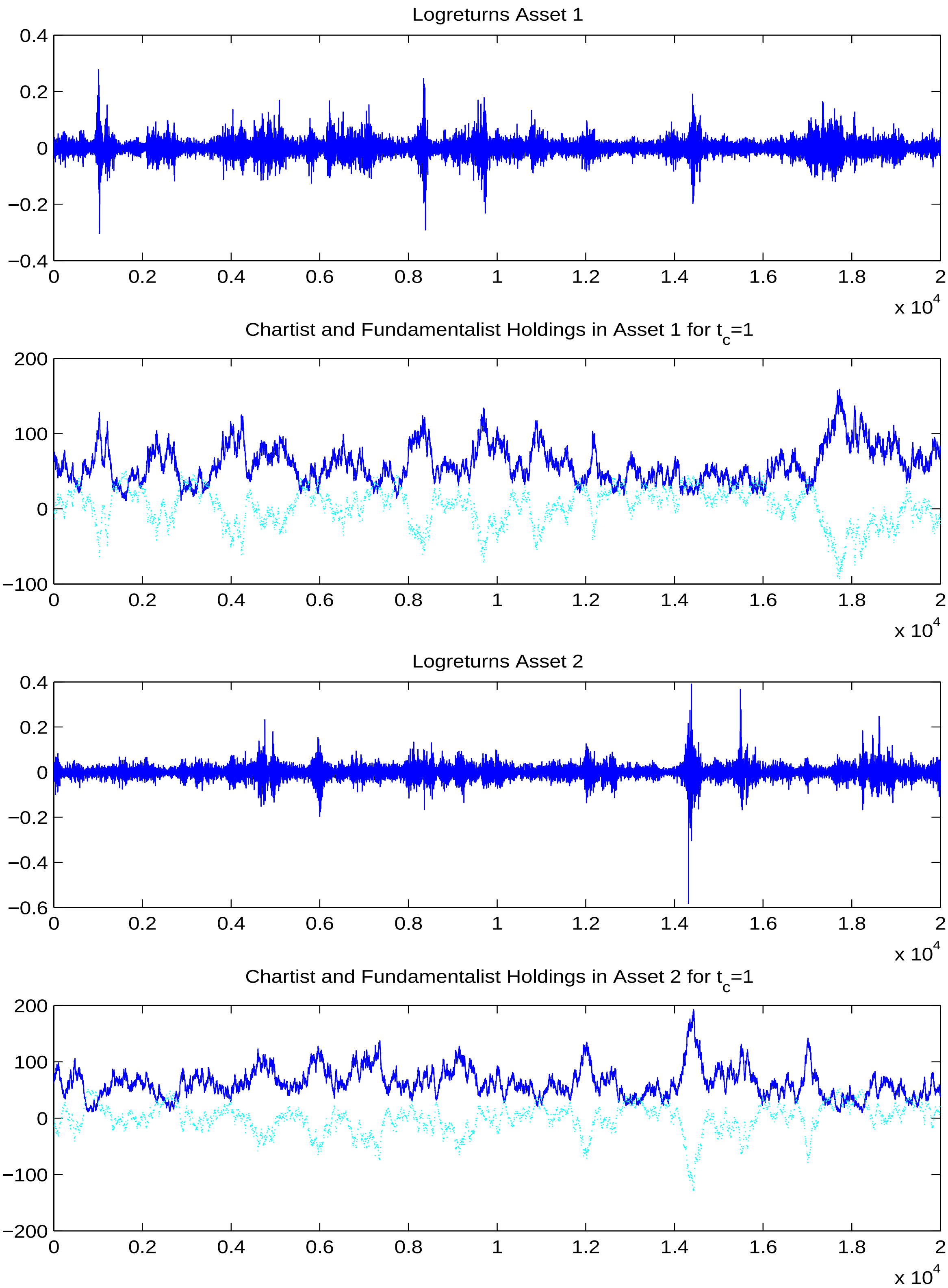

Figure 15. Logreturns and traders inventories in stock 1 (upper two panels) and stock 2 (lower two panels) for $t_c = 1$. The dark solid lines denote chartist holdings, and the light dotted lines fundamentalist holdings in the respective stock.

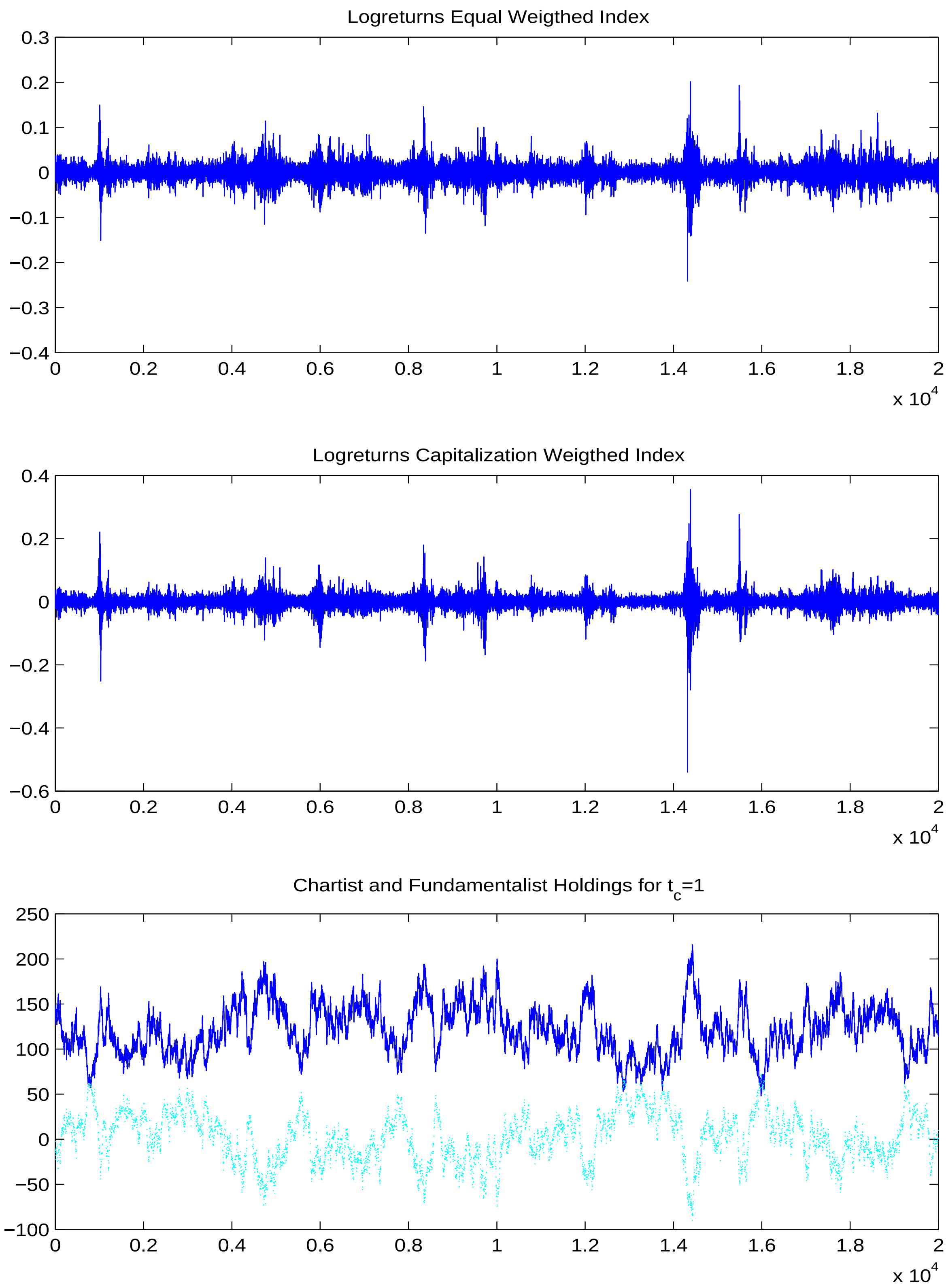

Figure 16. Index returns and traders inventories for $t_c = 1$. The dark solid lines denote chartist holdings, and the light dotted lines fundamentalist holdings aggregated over both stocks.

Table 17. Probability values of falsely rejecting $H_0: \ \rho \geq 1$ in Dickey-Fuller regressions including a constant for chartist holdings and without a constant for fundamentalist holdings using the one-sided critical values by MacKinnon (1996).

p-value	Stock 1	Stock 2	Aggregate
Chartist Holdings	$7.55 \cdot 10^{-11}$	$2.86 \cdot 10^{-10}$	$1.03 \cdot 10^{-16}$
Fundamentalist Holdings	$1.05 \cdot 10^{-10}$	$2.88 \cdot 10^{-10}$	$1.34 \cdot 10^{-16}$

study by Lux & Marchesi (2000).[140] The figures show chartists and fundamentalists holdings normalized at $t_c = 1$ (choosing any other value of t_c just changes the scale of the plot) together with the return series of the associated stock and the stock indices, respectively, for comparison. It can be seen that periods of high volatility tend to coincide with above average chartist positions. Volatility clusters occur typically for large chartist holdings because these usually coincide with only a few investors pursuing a fundamentalist strategy in the relevant stock, which was seen in equation (5.3) to speed up price changes.

Note that contrary to the simulations of the order-based setup of Lux & Marchesi discussed in section 4.3.2, the trader holdings are stationary in levels in this model, because traders holdings rather than orders have been linked to the level stationary mispricings and trader populations in equation (5.1). As is evident from table 17, the presence of a unit root is strongly rejected in all tests of any traders positions. This confirms our original conjecture that the risk of building up infinite inventories is not present in our model, consistent with the conservation of the number of shares discussed on page 118.

Consider finally the wealth dynamics for the two types of traders illustrated in figure 17. The upper panel contains the amount of cash aggregated by the chartist and fundamentalist subpopulations, whereas the lower panel includes also the market value of the stocks. It is immediately evident from both plots that chartists loose their money to fundamentalists, as was the case in the simulations of the model by Lux & Marchesi (2000). While one might again be tempted to conclude that chartist will go bankrupt and disappear, this need not necessarily be so for the same reasons as mentioned earlier

[140] see figures 1 to 4 in section 4.3.2.

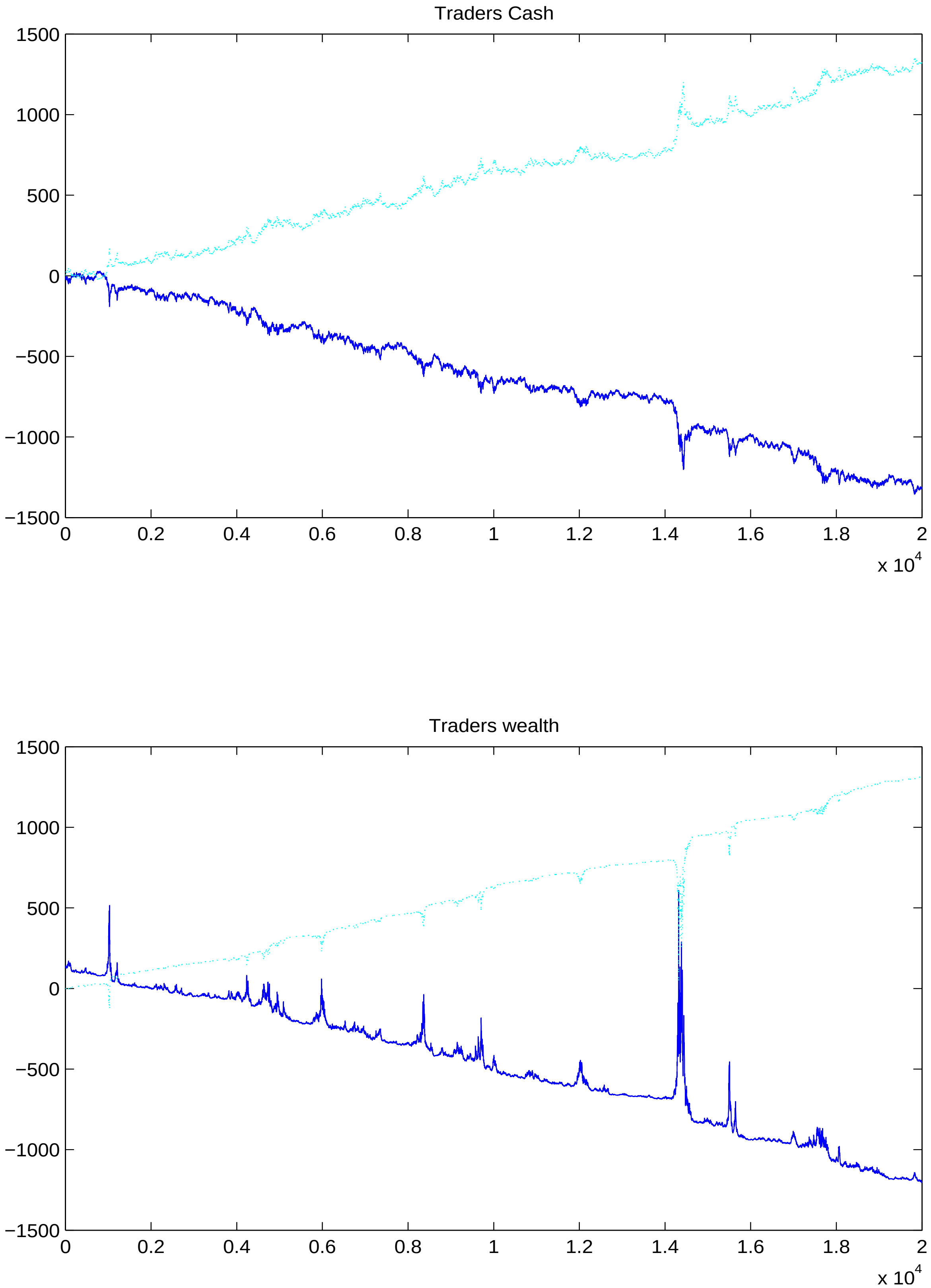

Figure 17. Aggregate cash and total wealth for chartists (dark solid line) and fundamentalists (light dotted line).

Table 18. Median estimates of the tail index over ten samples of 2,000 observations each and the range of estimates for common choices of the tail region.

Asset	2.5%tail			5%tail			10%tail		
	min	median	max	min	median	max	min	median	max
Stock 1	2.55	4.49	5.71	2.25	3.68	4.53	1.95	3.15	3.64
Stock 2	2.76	4.28	5.16	2.74	3.70	4.37	2.13	3.15	3.60
Index (EW)	2.68	4.41	5.58	2.89	3.82	4.81	2.27	3.18	3.66
Index (CW)	1.92	4.30	5.62	2.13	3.72	4.34	1.85	2.88	3.55

in the discussion of those simulations on page 113.

In the remainder of this section I shall demonstrate that the model of section 5.1 is capable of reproducing the stylized facts of financial returns in much the same way as Lux & Marchesi (2000), applying the same battery of tests to them as they did. Consider first the fat tail property. The kurtosis for the return series of the returns series of the individual stocks and the stock indices were already given in table 16. All of them were double digit numbers consistent with empirical findings.

As regards the tail index of the series, I follow Lux and Marchesi in splitting each of our datasets into 10 subsamples of 2,000 observations, and applying the Hill estimator with varying cut-off values to each of them, using again the algorithm of appendix A2. The results are presented in table 18. We find tail indices somewhere between 2 and 5 with increasing estimates for decreasing tail size, just like in their study and in harmony with empirical findings.[141]

Consider next the autocorrelation diagram of raw, squared and absolute returns for the two stocks and indices in figure 18. Similar to Lux & Marchesi and consistent with empirical findings, squared and absolute returns show much higher autocorrelations than raw returns with only minor fluctuations around zero.[142] Autocorrelation coefficients of absolute returns do not even decay to zero when considering 300 lags, which is consistent with both empirically observed data and long memory in return volatility.

[141] see table 2 on page 97 and section 2.4.
[142] see figure 6 on page 101 and section 2.6.

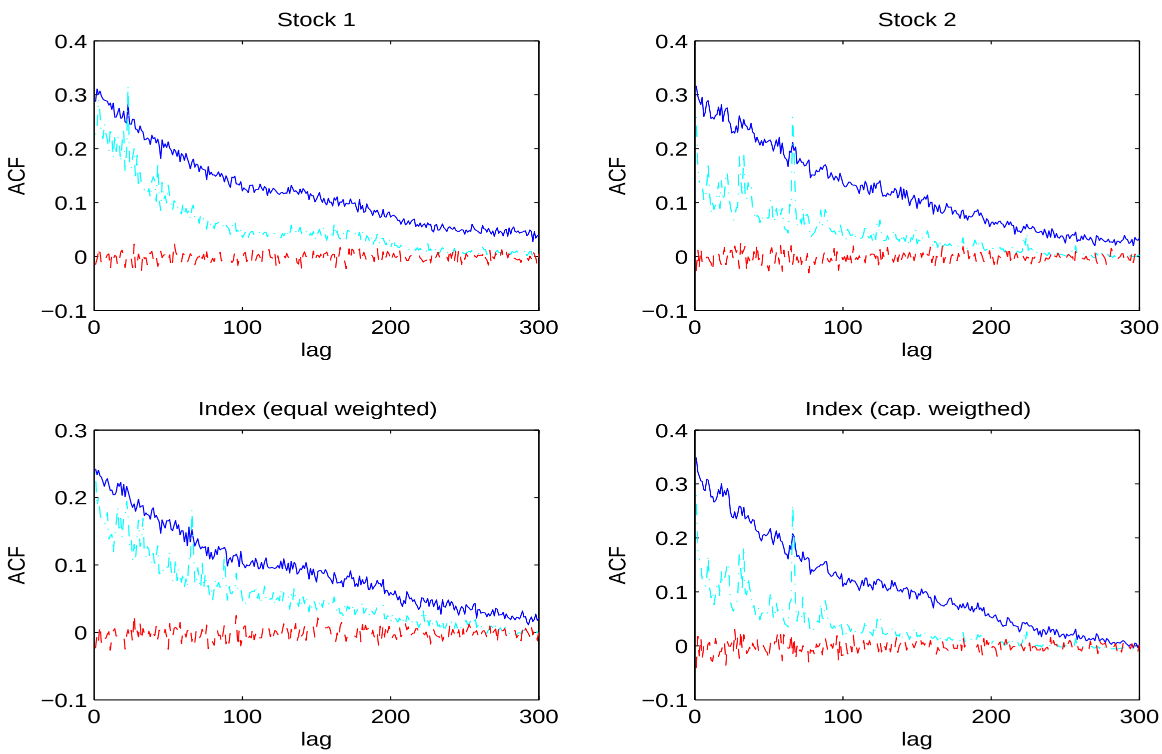

Figure 18. Autocorrelation diagram of absolute (dark solid line), squared (light dashed line) and raw returns (dark dashed line) over 300 lags.

Table 19. Estimates of the long memory parameter d for squared returns (upper panel) and absolute returns (lower panel) for both stocks, the equal weighted index, and the capitalization weighted index over the full sample of 20,000 observations and over 10 subsamples of 2,000 observations each. The last column contains the number of significantly positive estimated long memory parameters at a significance level of 5%. All estimates of d over the full sample are significantly positive.

Squared Returns	$\hat{d}$ full sample	min($\hat{d}$)	10 samples median($\hat{d}$)	max($\hat{d}$)	# $\hat{d}$ sign. > 0 in 10 samples
Stock 1	0.27	0.31	0.56	0.90	8
Stock 2	0.40	0.27	0.56	0.77	9
Index (EW)	0.35	0.21	0.51	0.74	8
Index (CW)	0.37	0.29	0.58	0.69	8
Absolute Returns	**$\hat{d}$ full sample**	**min($\hat{d}$)**	**10 samples median($\hat{d}$)**	**max($\hat{d}$)**	**# $\hat{d}$ sign. > 0 in 10 samples**
Stock 1	0.46	0.40	0.65	0.73	9
Stock 2	0.49	0.41	0.60	0.86	9
Index (EW)	0.48	0.22	0.55	0.74	7
Index (CW)	0.41	0.33	0.61	0.83	7

In order to test formally for long memory in return volatility, I follow again Lux & Marchesi in subdividing all datasets into 10 subsamples of 2,000 observations and applying the Geweke/Porter-Hudak estimator of the fractional differencing parameter d of a fractionally integrated ARMA model to each of them, using the algorithm presented in appendix A3. Similar to their study, the results presented in table 19 indicate evidence for long-term dependence with d estimated significantly larger than zero for most of the tests. Many of the estimates fall even into the region $d > 0.5$ indicating explosive volatility processes as in their study.[143] However, when using the full datasets, all estimates remain below 0.5 but significantly positive, as they should for stationary long memory processes.

[143]refer to tables 4 to 7 in section 4.3.2.

6 Conclusion

I have extended the univariate artificial market by Lux & Marchesi (2000) into a multivariate setup by including a second risky asset and a risk free bond. The order-based trading strategies of their model were replaced by corresponding position-based strategies in order to reconcile it better with the position-concerned trading behaviour in real markets and to avoid the unrealistic possibility of unlimited traders inventories. In order to add further realism to the model, asset allocation and security selection were modeled as two separate decision processes, in line with common practice in financial institutions.

The simulated return series of this artificial market share most of the stylized facts of financial returns. Serially uncorrelated returns with volatility clustering, leptokurtic return distributions with realistic tail indexes, and long memory in squared and absolute returns were all observed, both for the individual stocks and for the stock indexes. Assuming constant intrinsic values for both stocks, the individual stock returns were found to be close to cross-sectionally uncorrelated.

I have argued that the absence of positive cross-sectional correlations in simulated stock returns and unit-roots in their prices was solely due to the simplifying assumption of constant intrinsic values. As a side effect it was shown that the original claimed capacity of the model by Lux & Marchesi (2000) to produce unit-root prices even under the assumption of constant fundamental values, was probably due to a flawed application of the Dickey Fuller test. Therefore I have concluded that the unit-root property of prices like the positive correlation between stock returns should be attributed to economic fundamentals rather than behavioural effects from the interaction of traders. Future work will extend the results of this study to multiple stocks, whose intrinsic values follow unit root processes of varying correlation structure.

References

Abhyankar, A., L. S. Copeland & W. Wong (1995). Moment condition failure in high frequency financial data: Evidence from the S&P 500. *Applied Economics Letters* 2, 288–290.

Adler, R. J., R. E. Feldman & M. S. Taqqu (1998). *A Practical Guide to Heavy Tails.* Birkhäuser, Berlin.

Akgiray, V. & G. G. Booth (1987). Compound distribution models of stock returns: An empirical comparison. *Journal of Financial Research* 10:3, 269–280.

Alchian, A. A. (1950). Uncertainty, evolution, and economic theory. *Journal of Political Economy* 58, 211–221.

Alexander, S. S. (1961). Price movements in speculative markets: Trends or random walks? *Industrial Management Review* 2:2, 7–26. Reprinted in Cootner (1964), pages 199–218.

Alfarano, S. & T. Lux (2006). A noise trader model as a generator of apparent financial power laws and long memory. *Macroeconomic Dynamcis* , forthcoming.

Alfarano, S., T. Lux & F. Wagner (2005). Estimation of agent-based models: The case of an asymmetric herding model. *Computational Economics* 26, 19–49.

Allen, H. & M. P. Taylor (1990). Charts, noise and fundamentals in the London foreign exchange market. *Economic Journal* 100:400, 49–59.

Andersen, T. G. (1996). Return volatility and trading volume: An information flow interpretation of stochastic volatility. *Journal of Finance* 51:1, 169–204.

Andersen, T. G., T. Bollerslev, F. X. Diebold & H. Ebens (2001). The distribution of stock return volatility. *Journal of Financial Economics* 61:1, 43–76.

Ané, T. & H. Geman (2000). Order flow, transaction clock, and normality of asset returns. *Journal of Finance* 55:5, 2259–2284.

Ang, A. & G. Bekaert (2002). International asset allocation with regime shifts. *Review of Financial Studies* 15:4, 1137–1187.

Ang, A. & J. Chen (2002). Asymmetric correlations in equity portfolios. *Journal of Financial Economics* 63:3, 443–494.

Aparicio, F. M. & J. Estrada (2001). Empirical distributions of stock returns: European securities markets, 1990–95. *European Journal of Finance* 7, 1–21.

Ariel, R. A. (1987). A monthly effect in stock returns. *Journal of Financial Economics* 18:1, 161–174.

Ariel, R. A. (1990). High stock returns before holidays: Existence and evidence on possible causes. *Journal of Finance* 45:5, 1611–1626.

Arnéodo, A., J.-F. Muzy & D. Sornette (1998). "Direct" causal cascade in the stock market. *European Physical Journal B* 2, 277–282.

Arrow, K. J. (1982). Risk perception in psychology and economics. *Economics Inquiry* 20, 1–9.

Arthur, W. B. (1995). Complexity in economic and financial markets. *Complexity* 1, 20–25.

Ausloos, M. & K. Ivanova (2002). Multifractal nature of stock exchange prices. *Computer Physics Communications* 147, 582–585.

Bachelier, L. (1900). *Théorie de la Speculation*. Gauthier-Villars, Paris. Translated into English in Cootner (1964), pages 17–78.

Bacry, E., J. Delour & J. F. Muzy (2001). Multifractal random walk. *Physical Review E* 64:2, 026103–026106.

Bacry, E. & J. F. Muzy (2003). Log-infinitely divisible multifractal processes. *Communications in Mathematical Physics* 236, 449–475.

Badrinath, S. G. & S. Chatterjee (1988). On measuring skewness and elongation in common stock return distributions: The case of the market index. *Journal of Business* 61:4, 451–472.

Badrinath, S. G. & S. Chatterjee (1991). A data-analytic look at skewness and elongation in common stock return distributions. *Journal of Business & Economic Statistics* 9:2, 223–233.

Baillie, R. T. (1996). Long memory processes and fractional integration in econometrics. *Journal of Econometrics* 73, 5–59.

Baillie, R. T., T. Bollerslev & H. O. Mikkelsen (1996). Fractionally integrated generalized autoregressive conditional heteroskedasticiy. *Journal of Econometrics* 74, 3–30.

Ball, C. A. & W. N. Torous (1983). A simplified jump process for common stock returns. *Journal of Financial and Quantitative Analysis* 18:1, 53–65.

Ball, R., B. Lev & R. L. Watts (1976). Income variations and balance sheet compositions. *Journal of Accounting Research* 14:1, 1–9.

Banz, R. W. (1981). The relationship between return and market value of common stocks. *Journal of Financial Economics* 9:1, 3–18.

Barabási, A.-L. & T. Vicsek (1991). Multifractality of self-affine fractals. *Physical Review A* 44:4, 2730–2733.

Barber, B., R. Lehavy, M. McNichols & B. Trueman (2001). Can investors profit from the prophets? Security analyst recommendations and stock returns. *Journal of Finance* 56:2, 531–563.

Barndorff-Nielsen, O. E. (1977). Exponentially decreasing distributions for the logarithm of particle size. *Proceedings of the Royal Society of London Series A* 353, 401–419.

Barndorff-Nielsen, O. E. (1978). Hyperbolic distributions and distributions on a hyperbolae. *Scandinavian Journal of Statistics* 5, 151–157.

Barndorff-Nielsen, O. E. (1997). Normal inverse Gaussian distributions and stochastic volatility modelling. *Scandinavian Journal of Statistics* 24, 1–13.

Barndorff-Nielsen, O. E. (1998). Processes of normal inverse Gaussian type. *Finance and Stochastics* 2, 41–68.

Barndorff-Nielsen, O. E., E. Nicolato & N. Shepard (2002). Some recent developments in stochastic volatility modelling. *Quantitative Finance* 2:1, 11–23.

Barndorff-Nielsen, O. E. & N. Shepard (2001). Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *Journal of the Royal Statistical Society B* 63:2, 167–241.

Barndorff-Nielsen, O. E. & N. Shepard (2003). Integrated OU processes and non-Gaussian OU-based stochastic volatility models. *Scandinavian Journal of Statistics* 30, 277–295.

Barnea, A. & D. H. Downes (1973). A reexamination of the empirical distribution of stock price changes. *Journal of the American Statistical Association* 68:342, 348–350.

Basu, S. (1977). Investment performance of common stocks in relation to their price-earnings ratios: A test fo the efficient market hypothesis. *Journal of Finance* 32:3, 663–682.

Baumol, W. J. (1957). Speculation, profitability, and stability. *Review of Economics and Statistics* 39, 263–271.

Beja, A. & M. B. Goldman (1980). On the dynamic behavior of prices in disequilibrium. *Journal of Finance* 35:2, 235–248.

Bera, A. K. & M. L. Higgins (1993). Arch models: Properties, estimation and testing. *Journal of Economic Surveys* 7:4, 305–362.

Beran, J. (1994). *Statistics for Long-Memory Processes.* Chapman & Hall, New York.

Berthelsen, C. L., J. A. Glazier & S. Raghavachari (1994). Effective multifractal spectrum of a random walk. *Physical Review E* 49:3, 1860–1864.

Bikhchandani, S. & S. Sharma (2001). Herd behavior in financial markets: A review. *IMF Staff Papers* 47:3, 279–310.

Black, F. (1976). Studies of stock price volatility changes. In *1976 Proceedings from the American Statistical Association, Business and Economics Satistics Section*, pages 177–181, Washington DC.

Black, F. (1986). Noise. *Journal of Finance* 41:3, 529–543.

Black, F. (1993). Return and the beta. *Journal of Portfolio Management* 1993:3, 8–18.

Black, F. & M. Scholes (1973). The pricing of options and corporate liabilities. *Journal of Political Economy* 81:3, 637–654.

Blattberg, R. C. & N. J. Gonedes (1974). A comparison of the stable and student distributions as statistical models for stock prices. *Journal of Business* 47:2, 244–280.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics* 31:3, 307–327.

Bollerslev, T. (1987). A conditionally heteroskedastic time series model for speculative prices and rates of returns. *Reviw of Economics and Statistics* 69:3, 542–547.

Bollerslev, T., R. Y. Chou & K. F. Kroner (1992). ARCH modeling in finance. *Journal of Econometrics* 52, 5–59.

Bollerslev, T. & R. F. Engle (1993). Common persistence in conditional variances. *Econometrica* 61:1, 167–186.

Bollerslev, T., R. F. Engle & D. B. Nelson (1994). ARCH models. In Engle, R. F. & D. L. McFadden, editors, *Handbook of Econometrics*, volume 4.

Bollerslev, T. & D. Jubinski (1999). Equity trading and volatility: Latent information arrivals and common long-run dependencies. *Journal of Business and Economic Statistics* 17:1, 9–21.

Bookstaber, R. M. & J. B. McDonald (1987). A general distribution for describing security price returns. *Journal of Business* 60:3, 401–424.

Bossaerts, P. & P. Hillion (1999). Implementing statistical criteria to select return forecasting models. *Review of Financial Studies* 12:2, 405–428.

Bouchaud, J.-P., A. Matacz & M. Potters (2001). Leverage effect in financial markets: The retarded volatility model. *Physical Review Letters* 87:22, 228701–(1–4).

Bouchaud, J.-P. & M. Potters (2001). More stylized facts of financial markets: Leverage effect and downside correlations. *Physica A* 299, 60–70.

Bouchaud, J.-P., M. Potters & M. Meyer (2000). Apparent multifractality in financial time series. *European Physical Journal B* 13, 595–599.

Box, G. E. P. & G. C. Tiao (1973). *Baysian Inference in Statistical Analysis.* Addison-Wesley, Reading, Massachusetts.

Boyer, B. H., M. S. Gibson & M. Loretan (1999). Pitfalls in tests for changes in correlations. International Finance Discussion Paper 597, Board of Governors of the Federal Reserve System, Washington DC.

Braun, P. A., D. B. Nelson & A. M. Sunier (1995). Good news, bad news, volatility, and betas. *Journal of Finance* 50:5, 1575–1603.

Breen, W. J. & R. A. Korajczyk (1995). On selection biases in book-to-market based tests of asset pricing models. Working Paper 167, Kellog Graduate Shool of Management, Northwestern University.

Breidt, F. J., N. Crato & P. de Lima (1998). The detecetion and estimation of long memory in stochastic volatility. *Journal of Econometrics* 83, 325–348.

Brennan, M. J., T. Chordia & A. Subrahmanyam (1998). Alternative factor specifications, security characteristics and the cross-section of expected stock returns. *Journal of Financial Economics* 49, 345–373.

Breymann, W., S. Ghashghaie & P. Talkner (2000). A stochastic cascade model for fx dynamics. *International Journal of Theoretical and Applied Finance* 3:3, 357–360.

Brock, W., W. Dechert & J. Scheinkman (1987). A test for independence based on the correlation dimension. Working Paper 9520, University of Wisconsin at Madison.

Brock, W. A. & C. H. Hommes (1997). A rational route to randomness. *Econometrica* 65:5, 1059–1095.

Brock, W. A. & C. H. Hommes (1998). Heterogeneous beliefs and routes to chaos in a simple asset pricing model. *Journal of Economic Dynamics and Control* 22, 1235–1274.

Brock, W. A., D. A. Hsieh & B. LeBaron (1991). *Nonlinear Dynamics, Chaos, and Instability.* MIT Press, Cambridge.

Brock, W. A., J. Lakonishok & B. LeBaron (1992). Simple technical trading rules and the stochastic properties of stock returns. *Journal of Finance* 47:5, 1731–1764.

Butler, R. J., J. B. McDonald, R. D. Nelson & S. B. White (1990). Robust and partially adaptive estimation of regression models. *Review of Economics and Statistics* 72:2, 321–327.

Caginalp, G. & H. Laurent (1998). The predictive power of price patterns. *Applied Mathematical Finance* 5, 181–205.

Calvet, L. & A. Fisher (2001). Forecasting multifractal volatility. *Journal of Econometrics* 105, 27–58.

Calvet, L. & A. Fisher (2002). Multifractality in asset returns: Theory and evidence. *Review of Economics and Statistics* 84:3, 381–406.

Calvet, L. & A. Fisher (2003). Regime-switching and the estimation of multifractal processes. Harvard Institute of Economic Research Discussion Paper 1999, Harvard University, Cambridge, Massachusetts.

Campbell, J. Y. & L. Hentschel (1992). No news is good news. *Journal of Financial Economics* 31, 281–318.

Campbell, J. Y., A. W. Lo & A. C. MacKinlay (1997). *The Econometrics of Financial Markets*. Princeton University Press, Princeton, New Jersey.

Caparrelli, F., A. M. D'Arcangelis & A. Cassuto (2004). Herding in the italian stock market: A case of behavioral finance. *Journal of Behavioral Finance* 5:4, 222–230.

Carr, P., H. Geman, D. B. Madan & M. Yor (2002). The fine structure of asset returns: An empirical investigation. *Journal of Business* 75:2, 305–332.

Carr, P., H. Geman, D. B. Madan & M. Yor (2003). Stochastic volatility for Lévy processes. *Mathematical Finance* 13:3, 345–382.

Chan, L. K. C., N. Jegadeesh & J. Lakonishok (1996). Momentum strategies. *Journal of Finance* 51:5, 1681–1713.

Chen, J., H. Hong & J. C. Stein (2001). Forecasting crashes: Trading volume, past returns, and conditional skewness in stock prices. *Journal of Financial Economics* 61, 345–381.

Cheung, Y.-W., M. D. Chinn & I. W. Marsh (2000). How do UK-based foreign exchange dealers think their market operates? NBER Working Paper 7524, National Bureau of Economic Research.

Cheung, Y.-W. & L. K. Ng (1992). Stock price dynamics and firm size: An empirical investigation. *Journal of Finance* 47:5, 1985–1997.

Chiarella, C. (1992). The dynamics of speculative behaviour. *Annals of Operations Research* 37, 101–123.

Chiarella, C., R. Dieci & X.-Z. He (2001). Asset price and wealth dynamics under heterogeneous expectations. *Quantitative Finance* 1:5, 509–526.

Chiarella, C., R. Dieci & X.-Z. He (2003). Heterogeneous beliefs, risk and learning in a simple asset pricing model with a market maker. *Macroeconomic Dynamics* 7:4, 503–536.

Chiarella, C., R. Dieci & X.-Z. He (2006). Heterogeneous expectations and speculative behaviour in a dynamic multi-asset framework. *Journal of Economic Behavior and Organization* , forthcoming.

Chopra, N., J. Lakonishok & J. R. Ritter (1992). Measuring abnormal performance: Do stocks overreact? *Journal of Financial Economics* 31:2, 235–268.

Christie, A. A. (1982). The stochastic behavior of common stock variances. *Journal of Financial Economics* 10, 407–432.

Chung, C.-F. (1999). Estimating the fractionally integrated GARCH model. Working paper, National Taiwan University, Taipei, TW.

Cizeau, P., Y. Liu, M. Meyer, C.-K. Peng & H. E. Stanley (1997). Volatility distribution in the S&P500 stock index. *Physica A* 245, 441–445.

Cizeau, P., M. Potters & J.-P. Bouchaud (2001). Correlation structure of extreme stock returns. *Quantitative Finance* 1:2, 217–223.

Clark, P. K. (1973). A subordinated stochastic process model with finite variance for speculative prices. *Econometrica* 41:1, 135–155.

Cont, R. (2001). Empirical properties of asset returns: stylized facts and statistical issues. *Quantitative Finance* 1:2, 223–236.

Cont, R. & J.-P. Bouchaud (2000). Herd behavior and aggregate fluctuations in financial markets. *Macroeconomic Dynamics* 4, 170–196.

Cootner, P. F. (1964). *The Random Character of Stock Market Prices.* MIT Press, Cambridge, Massachusetts.

Crato, N. & P. J. de Lima (1994). Long-range dependence in the conditional variance of stock returns. *Economics Letters* 45, 281–285.

Cross, F. (1973). The behavior of stock prices on fridays and mondays. *Financial Analysts Journal* 1973:6, 67–69.

Cutler, D. M., J. M. Poterba & L. H. Summers (1989). What moves stock prices? *Journal of Portfolio Management* 1989:1, 67–69.

Das, S. R. & R. Uppal (2004). Systemic risk and international portfolio choice. *Journal of Finance* 59:6, 2809–2834.

Davis, E. P. (2003). Institutional investors, financial market efficiency and financial stability. Discusssion Paper PI-0303, The Pensions Institute, University of London.

Davis, E. P. & B. Steil (2001). *Institutional Investors.* MIT Press, Cambridge, Massachusetts.

Davis, J. L., E. F. Fama & K. R. French (2000). Characteristics, covariances, and average returns: 1929 to 1997. *Journal of Finance* 55:1, 389–406.

Day, R. H. & W. Huang (1990). Bulls, bears and market sheep. *Journal of Economic Behavior and Organization* 14, 299–329.

De Bondt, W. F. M. & R. Thaler (1985). Does the stock market overreact? *Journal of Finance* 40:3, 793–805.

de Haan, L., S. I. Resnick, H. Rootzen & C. G. de Vries (1989). Extremal behaviour of solutions to a stochastic difference equation with applications to ARCH processes. *Stochastic Processes and their Applications* 32, 213–224.

Degiannakis, S. & E. Xekalaki (2004). Autoregressive conditional heteroscedasticity (ARCH) models: A review. *Qualitative Technology and Quantitative Management* 1:2, 271–324.

DeGrauwe, P., H. Dewachter & M. Embrechts (1993). *Exchange Rate Theory: Chaotic Models of the Foreign Exchange Market.* Blackwell, Oxford.

Delbaen, F. & W. Schachermayer (1994). A general version of the fundamental theorem of asset pricing. *Mathematische Annalen* 300, 463–520.

DeLong, J. B., A. Shleifer, L. H. Summers & R. J. Waldman (1990). Noise trader risk in financial markets. *Journal of Political Economy* 98, 703–738.

Diebold, F. X. (1988). *Empirical Modeling of Exchange Rate Dynamics*, volume 303 of *Lecture Notes in Economics and Mathematical Systems.* Springer, Berlin.

Diebold, F. X. & A. Inoue (2001). Long memory and regime switching. *Journal of Econometrics* 105, 131–159.

Ding, Z. & C. W. Granger (1996). Modeling volatility persistence of speculative returns: A new approach. *Journal of Econometrics* 73, 185–215.

Ding, Z., C. W. Granger & R. F. Engle (1993). A long memory property of stock market returns and a new model. *Journal of Empirical Finance* 1, 83–106.

Drost, F. C. & T. E. Nijman (1993). Temporal aggregation of Garch processes. *Econometrica* 61:4, 909–927.

Duffee, G. R. (1995). Stock return and volatility: A firm-level analysis. *Journal of Financial Economics* 37, 399–420.

DuMouchel, W. H. (1973). Stable distributions in statistical inference: 1. symmetric stable distributions compared to other symmetric long-tailed distributions. *Journal of the American Statistical Association* 68:342, 469–477.

Easley, D. & M. O'Hara (1992). Time and the process of security price adjustment. *Journal of Finance* 47:2, 577–605.

Eberlein, E. & U. Keller (1995). Hyperbolic distributions in finance. *Bernoulli* 1:3, 281–299.

Embrechts, P., C. Klüppelberg & T. Mikosch (1997). *Modelling Extremal Events.* Springer, Berlin.

Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica* 50:4, 987–1008.

Engle, R. F. & T. Bollerslev (1986). Modelling the persistence of conditional variances. *Econometric Reviews* 5:1, 1–50.

Engle, R. F. & G. G. Lee (1993). A permanent and transitory component model of stock return volatility. Discussion Paper 92-44R, University of California, San Diego.

Engle, R. F., D. M. Lilien & R. P. Robins (1987). Estimating time varying risk premia in the term structure: The Arch-M model. *Econometrica* 55:2, 391–407.

Epps, T. W. (1975). Security price changes and transaction volumes: Theory and evidence. *American Economic Review* 65:4, 586–597.

Epps, T. W. & M. L. Epps (1976). The stochastic dependence of security price changes and transaction volumes: Implications for the mixture-of-distributions hypothesis. *Econometrica* 44:2, 305–321.

Erb, C. B., C. R. Harvey & T. E. Viskanta (1994). Forecasting international equity correlations. *Financial Analysts Journal* 1994:6, 32–45.

Evans, G. W. & S. Honkapohja (2001). *Learning and Expectations in Macroeconomics.* Princeton University Press, Princeton, NJ.

Evertsz, C. J. G. & B. B. Mandelbrot (1992). Multifractal measures. In Pleitgen, H.-O., H. Jürgens & D. Saupe, editors, *Chaos and Fractals*, pages 921–954, Berlin. Springer.

Fama, E. F. (1963). Mandelbrot and the stable Paretian hypothesis. *Journal of Business* 36:4, 420–429.

Fama, E. F. (1965). The behaviour of stock-market prices. *Journal of Business* 38:1, 34–105.

Fama, E. F. (1970). Efficient capital markets. *Journal of Finance* 25:2, 383–417.

Fama, E. F. (1976). *Foundations of Finance.* Basil Blackwell, Oxford.

Fama, E. F. (1981). Stock returns, real activity, inflation, and money. *American Economic Review* 71:4, 545–565.

Fama, E. F. (1998). Market efficiency, long-term returns and behavioral finance. *Journal of Financial Economics* 49, 283–306.

Fama, E. F. & K. R. French (1992). The cross-section of expected stock returns. *Journal of Finance* 47:2, 427–465.

Fama, E. F. & K. R. French (1993). Common risk factors in the returns of stocks and bonds. *Journal of Financial Economics* 33:1, 3–56.

Fama, E. F. & K. R. French (1995). Size and book-to-market factors in earnings and returns. *Journal of Finance* 50:1, 131–155.

Fama, E. F. & K. R. French (1996). Multifactor explanations of asset pricing anomalies. *Journal of Finance* 51:1, 55–84.

Fama, E. F. & K. R. French (1998). Value versus growth: The international evidence. *Journal of Finance* 53:6, 1975–1999.

Farmer, J. D. (2000). Physicists attempt to scale the ivory towers of finance. *International Journal of Theoretical and Applied Finance* 3:3, 311–333.

Farmer, J. D. (2002). Market force, ecology, and evolution. *Industrial and Corporate Change* 11:5, 895–953.

Farmer, J. D., L. Gillemot, F. Lillo, S. Mike & A. Sen (2004). What really causes large price changes. *Quantitative Finance* 4:4, 383–397.

Farmer, J. D. & S. Joshi (2002). The price dynamics of common trading strategies. *Journal of Economic Behavior and Organization* 49, 149–171.

Farrel, M. J. (1966). Profitable speculation. *Economica* 33, 183–193.

Feller, W. (1966). *An Introduction to Probability Theory and Its Applications*, volume II. John Wiley & Sons, New York.

Fielitz, B. D. (1971). Stationarity of random data: Some implications for the distribution of stock price changes. *Journal of Financial and Quantitative Analysis* 6:3, 1025–1034.

Fielitz, B. D. & J. P. Rozelle (1983). Stable distributions and the mixtures of distributions hypotheses for common stock returns. *Journal of the American Statistical Association* 78:381, 28–36.

Figlewski, S. & X. Wang (2000). Is the "leverage effect" a leverage effect? Working Paper FIN-00-037, Stern School of Business, New York University.

Fisher, R. A. & L. H. C. Tippett (1928). Limiting forms of the frequency distribution of the largest or smallest number of the sample. In *Proceedings of the Cambridge Philosophical Society*, volume 24, pages 180–190.

Föllmer, H. (1974). Random economies with many interacting agents. *Journal of Mathematical Economics* 1, 51–62.

Forbes, K. F. & R. Rigobon (2002). No contagion, only independence: Measuring stock market comovements. *Journal of Finance* 57:5, 2223–2261.

Frankel, J. A. & K. A. Froot (1987a). Short-term and long-term expectations of the yen/dollar exchange rate: Evidence from survey data. *Journal of the Japanese and International Economies* 1, 249–274.

Frankel, J. A. & K. A. Froot (1987b). Using survey data to test standard propositions regarding exchange rate expectations. *American Economic Review* 77, 133–153.

Frankel, J. A. & K. A. Froot (1990a). Chartists, fundamentalists, and the demand for dollars. In Courakis, A. & M. Taylor, editors, *Private Behavior and Government Policy in Interdependent Economies*, pages 73–128, Oxford. Clarendon Press.

Frankel, J. A. & K. A. Froot (1990b). Chartists, fundamentalists, and trading in the foreign exchange market. *American Economic Review* 80:2, 181–185.

French, K. R. (1980). Stock returns and the weekend effect. *Journal of Financial Economics* 8:1, 55–69.

French, K. R., G. W. Schwert & R. F. Stambaugh (1987). Expected stock returns and volatility. *Journal of Financial Economics* 19, 3–29.

Friedman, B. M. & D. I. Laibson (1989). Economic implications of extraordinary movements in stock prices. *Brookings Papers on Economic Activity* 2, 137–189.

Friedman, M. (1953). The case for flexible exchange rates. In Friedman, M., editor, *Essays in Positive Economics*, pages 157–203, Chicago. Chicago University Press. Reprinted in MacDonald & Taylor (1992), pages 3–49.

Fuller, W. A. (1976). *Introduction to Statistical Time Series.* Wiley, New York.

Gallant, A. R., P. E. Rossi & G. Tauchen (1992). Stock prices and volume. *Review of Financial Studies* 5:2, 199–242.

Gallant, A. R., P. E. Rossi & G. Tauchen (1993). Nonlinear dynamic structures. *Econometrica* 61:4, 871–907.

Gandolfo, G. (1996). *Economic Dynamics.* Springer, Berlin, 3rd edition.

Gaunersdorfer, A. (2001). Adaptive beliefs and the volatility of asset prices. *Central European Journal of Operations Research* 9, 5–30.

Gaunersdorfer, A. & C. H. Hommes (2007). A nonlinear structural model for volatility clustering. In Kirman, A. P. & G. Teyssière, editors, *Long Memory in Economics*, pages 265–288. Springer.

Geanakoplos, J. (1992). Common knowledge. *Journal of Economic Perspectives* 6, 52–83.

Geman, H. (2002). Pure jump Lévy processes for asset price modelling. *Journal of Banking and Finance* 26, 1297–1316.

Geman, H., D. B. Madan & M. Yor (2001). Time changes for Lévy processes. *Mathematical Finance* 11:1, 79–96.

Geweke, J. & S. Porter-Hudak (1983). The estimation and application of long memory time series models. *Journal of Time Series Analysis* 4:4, 221–238.

Ghashghaie, S., W. Breymann, J. Peinke, P. Talkner & Y. Dodge (1996). Turbulent cascades in foreign exchange markets. *Nature* 381, 767–770.

Ghysels, E., A. C. Harvey & E. Renault (1996). Stochastic volatility. In Maddala, G. S. & C. R. Rao, editors, *Handbook of Statistics*, volume 14, pages 119–192, Amsterdam. Elsevier Science B.V.

Gillemot, L., J. D. Farmer & F. Lillo (2006). There's more to volatility than volume. *Quantitative Finance* 6:5, 371–384.

Gillemot, L., J. Töli, J. Kertesz & K. Kaski (2000). Time-independent models of asset returns revisited. *Physica A* 282, 304–324.

Gilli, M. & P. Winker (2003). A global optimization heuristic for estimating agent based models. *Computational Statistics and Data Analysis* 42, 299–312.

Giraitis, L., P. Kokoszka & R. Leipus (2000). Stationary ARCH models: Dependence structure and central limit theorem. *Econometric Theory* 16:1, 3–22.

Glosten, L. R., R. Jaganathan & D. E. Runkle (1993). On the relation between the expected value and the volatility of the nominal excess return on stocks. *Journal of Finance* 48:5, 1779–1801.

Gopikrishnan, P., M. Meyer, L. A. N. Amaral & H. E. Stanley (1998). Inverse cubic law for the distribution of stock price variations. *European Physical Jornal B* 3, 139–140.

Gopikrishnan, P., V. Plerou, L. A. N. Amaral, M. Meyer & H. E. Stanley (1999). Scaling of the distribution of fluctuations of financial market indices. *Physical Review E* 60, 5305–5316.

Gouriéroux, C. (1997). *ARCH Models and Financial Applications*. Springer, Berlin.

Granger, C. W. J. & R. Joyeux (1980). An introduction to long-memory time series models and fractional differencing. *Journal of Time Series Analysis* 1:1, 15–29.

Granger, C. W. J. & T. Teräsvirta (1999). A simple nonlinear time series model with misleading linear properties. *Economics Letters* 62, 161–165.

Grau-Carles, P. (2000). Empirical evidence of long-range correlations in stock returns. *Physica A* 287, 396–404.

Gray, J. B. & D. W. French (1990). Empirical comparison of distributional models for stock index returns. *Journal of Business Finance and Accounting* 17:3, 451–459.

Grossman, S. J. & J. E. Stiglitz (1980). On the impossibility of informationally efficient markets. *American Economic Review* 70, 393–408.

Hagerman, R. L. (1978). More evidence on the distribution of security returns. *Journal of Finance* 33:4, 1213–1221.

Hakansson, N., J. Kunkel & J. Ohleson (1982). Suffient and necessary conditions for information to have social value in pure exchange. *Journal of Finance* 37:5, 1169–1181.

Haken, H. (1983). *Synergetics: An Introduction*, volume 1 of *Springer Series in Synergetics*. Springer, Berlin, 3 edition.

Hansen, L. P. (1982). Large sample properties of generalized method of moments estimators. *Econometrica* 50:4, 1029–1054.

Harris, L. (1986). Cross-security tests of the mixture of distributions hypothesis. *Journal of Financial and Quantitative Analysis* 21:1, 39–46.

Harris, R. D. F. & C. C. Küçüközmen (2001). The empirical distributions of UK and US stock returns. *Journal of Business Finance and Accounting* 28:5, 715–739.

Haugen, R. A. & N. L. Baker (1996). Commonality in the determinants of expected stock returns. *Journal of Financial Economics* 41, 401–439.

Haugen, R. A., E. Talmor & W. N. Torous (1991). The effect of volatility changes on the level of stock prices and subsequent expected returns. *Journal of Finance* 46, 985–1007.

Hawanini, G. & D. B. Keim (1995). On the predictability of common stock returns: World-wide evidence. In Jarrow, R., V. Maksimovic & W. Ziemba, editors, *Handbooks in Operations Research and Management Science*, volume 9: Finance, pages 497–544, Amsterdam. North-Holland.

Heiner, R. A. (1983). The origin of predictable behavior. *American Economic Review* 73, 560–595.

Heston, S. L. (1993). A closed form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies* 6:2, 327–343.

Hill, B. M. (1975). A simple general approach to inference about the tail of a distribution. *Annals of Statistics* 3:5, 1163–1174.

Hirshleifer, D. & S. H. Teoh (2003). Herd behaviour and cascading in capital markets: A review and synthesis. *European Financial Management* 9:1, 25–66.

Hogan, S., R. Jarrow, M. Teo & M. Warachka (2004). Testing market efficiency using statistical arbitrage with applications to momentum and value strategies. *Journal of Financial Economics* 73, 525–565.

Honkapohja, S. & T. Ito (1983). Stability with regime switching. *Journal of Economic Theory* 29:1, 22–48.

Hosking, J. R. M. (1981). Fractional differencing. *Biometrika* 68:1, 165–176.

Hsieh, D. A. (1989). Modeling heteroscedasticity in daily foreign exchange rates. *Journal of Business and Economic Statistics* 7, 307–317.

Hsieh, D. A. (1991). Chaos and nonlinear dynamics: Application to financial markets. *Journal of Finance* 46:5, 1839–1877.

Hsu, D.-A. (1977). Tests for variance shift at an unknown time point. *Applied Statistics* 26:3, 279–284.

Hsu, D.-A. (1979a). Detecting shifts of parameter in gamma sequences with application to stock price and air traffic flow analysis. *Journal of the American Statistical Association* 74:365, 31–40.

Hsu, D.-A. (1979b). Long-tailed distributions for position errors in navigation. *Journal of the Royal Statistical Society Series C (Applied Statistics)* 28:1, 62–72.

Hsu, D.-A. (1980). Further analyses of position errors in navigation. *Journal of Navigation* 33, 452–474.

Hsu, D.-A. (1982). A Bayesian robust detection of shift in the risk structure of stock market returns. *Journal of the American Statistical Association* 77:377, 29–39.

Hsu, D.-A., R. B. Miller & D. W. Wichern (1974). On the stable Paretian behavior of stock-market prices. *Journal of the American Statistical Association* 69:345, 108–113.

Hull, J. & A. White (1987). The pricing of options on assets with stochastic volatilities. *Journal of Finance* 42:2, 281–300.

Hurst, H. E. (1951). Long-term storage capacity of reservoirs. *Trans. of the Am. Soc. of Civil Engineers* 116, 770–808.

Hwang, S. & M. Salmon (2004). Market stress and herding. *Journal of Empirical Finance* 11, 585–616.

Iori, G. (2002). A microsimulation of traders activity in the stock market: The role of heterogeneity, agents' interactions and trade frictions. *Journal of Economic Behavior and Organization* 49, 269–285.

Ivanova, K. & M. Ausloos (1999). Low q-moment multifractal analysis of gold price, Dow Jones Industrial Average and BGL-USD exchange rate. *European Physical Journal B* 8, 665–669.

Jansen, D. W. & C. G. de Vries (1991). On the frequency of large stock returns: Putting boosts and busts into perspective. *Review of Economics & Statistics* 73, 18–24.

Jegadeesh, N. & S. Titman (1993). Returns to buying winners and selling losers: Implications for stock market efficiency. *Journal of Finance* 48:1, 65–91.

Jung, R. C. & R. Liesenfeld (1996). Testing the bivariate mixture hypothesis using German stock market data. *European Financial Management* 2:3, 273–297.

Kachitvichyanukul, V. & B. W. Schmeiser (1988). Binomial random variate generation. *Communications of the ACM* 31:2, 216–222.

Kahneman, D. (2003). Maps of bounded rationality: Psychology for behavioral economics. *American Economic Review* 93:5, 1449–1475.

Kahneman, D. & A. Tversky (1973). On the psychology of prediction. *Psychological Review* 80:4, 237–251.

Kaldor, N. (1939). Speculation and economic stability. *Review of Economic Studies* 7:1, 1–27.

Kaldor, N. (1961). Capital accumulation and economic growth. In Lutz, F. A. & D. C. Hague, editors, *The Theory of Capital*, pages 177–222, London. MacMillan.

Karpoff, J. M. (1987). The relation between price changes and trading volume: A survey. *Journal of Financial and Quantitative Analysis* 22:1, 109–126.

Keim, D. B. (1983). Size-related anomalies and stock return seasonality: Further empirical evidence. *Journal of Financial Economics* 12:1, 13–32.

Kendall, M. G. & A. B. Hill (1953). The analysis of economic time-series. *Journal of the Royal Statistical Society Series A* 116:1, 11–34.

Keynes, J. M. (1936). *The General Theory of Employment, Interest and Money.* Harcourt, Brace and Company, New York.

Khintchine, A. Y. (1937). Zur Theorie der unbeschränkt teilbaren Verteilungsgesetze. *Rec. Math. [Mat. Sbornik] N. S.* 2, 79–120.

Kim, D. & S. J. Kon (1994). Alternative models for the conditional heteroscedasticity of stock returns. *Journal of Business* 67:4, 563–598.

Kim, T.-H. & H. White (2004). On more robust estimation of skewness and kurtosis. *Finance Research Letters* 1, 56–73.

King, M. A. & S. Wadhwani (1990). Transmission of volatility between stock markets. *Review of Financial Studies* 3:1, 5–33.

King, W. I. (1930). *Index Numbers Elucidated.* referenced in Working (1934).

Kirman, A. P. (1991). Epidemics of opinion and speculative bubbles in financial markets. In Taylor, M., editor, *Money and Financial markets*, pages 354–367, London. Macmillan.

Kirman, A. P. (1992). Whom or what does the representative individual represent? *Journal of Economic Perspectives* 6:2, 117–136.

Kirman, A. P. (1993). Ants, rationality and recruitment. *Quaterly Journal of Economics* 108:1, 137–156.

Kirman, A. P. & G. Teyssière (2002). Microeconomic models for long memory in the volatility of financial time series. *Studies in Nonlinear Dynamics and Econometrics* 5:4, 281–302.

Kon, S. J. (1984). Models of stock returns—a comparison. *Journal of Finance* 39:1, 147–165.

Koponen, I. (1995). Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process. *Physical Review E* 52:1, 1197–1199.

Kothari, S. P., J. Shanken & R. G. Sloan (1995). Another look at the cross-section of expected stock returns. *Journal of Finance* 50:1, 185–224.

Koutmos, G., U. Lee & P. Theodossiou (1994). Persistence in variance, structural change, and the GARCH model. *Journal of Economics and Business* 46:2, 101–112.

Lakonishok, J., A. Shleifer & R. W. Vishny (1994). Contrarian investment, extrapolation, and risk. *Journal of Finance* 49:5, 1541–1578.

Lakonishok, J. & S. Smidt (1988). Are seasonal anomalies real? A ninety-year perspective. *Review of Financial Studies* 1:4, 403–425.

Lamoureux, C. G. & W. D. Lastrapes (1990). Persistence in variance, structural change, and the GARCH model. *Journal of Business & Economic Statistics* 8, 225–234.

Lau, A. H.-L., H.-S. Lau & J. R. Wingender (1990). The distribution of stock returns: New evidence against the stable model. *Journal of Business & Economic Statistics* 8:2, 217–223.

LeBaron, B. (2001). Stochastic volatility as a simple generator of apparent financial power laws and long memory. *Quantitative Finance* 1:6, 621–631.

Lee, S. B. & K. J. Kim (1993). Does the october 1987 crash strengthen the co-movements among national stock markets? *Review of Financial Economics* 3:1/2, 89–102.

LeRoy, S. F. (1989). Efficient capital markets and martingales. *Journal of Economic Literature* 27:4, 1583–1621.

Lévy, P. (1925). *Calcul des probabilités.* Gauthier-Villars, Paris.

Liesenfeld, R. (1998). Dynamic bivariate mixture models: Modeling the behavior of prices and trading volume. *Journal of Business &Economic Statistics* 16:1, 101–109.

Lintner, J. (1965a). Security prices, risk, and maximal gains from diversification. *Journal of Finance* 20:4, 587–616.

Lintner, J. (1965b). The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *Review of Economics and Statistics* 47:1, 13–37.

Litvinova, J. (2003). Volatility asymmetry in high frequency data. manuscript, Duke Uiversity.

Liu, Y., P. Cizeau, M. Meyer, C.-K. Peng & H. E. Stanley (1997). Correlations in conomic time series. *Physica A* 245, 437–440.

Liu, Y., P. Gopikrishnan, P. Cizeau, M. Meyer, C.-K. Peng & H. E. Stanley (1999). Statistical properties of the volatility of price fluctuations. *Physical Review E* 60:2, 1390–1400.

Lo, A. W. (1991). Long-term memory in stock market prices. *Econometrica* 59:5, 1279–1313.

Lo, A. W. & A. C. MacKinlay (1990). Data-snooping biases in tests of financial asset pricing models. *Review of Financial Studies* 3:3, 431–467.

Lobato, I. N. & N. E. Savin (1996). Real and spurious long-memory properties of stock-market data. *Journal of Business &Economic Statistics* 16:3, 261–268.

Lobato, I. N. & C. Velasco (2000). Long memory in stock-market trading volume. *Journal of Business &Economic Statistics* 18:4, 410–427.

Longin, F. & B. Solnik (1995). Is the correlation in international equity returns constant: 1960–1990? *Journal of International Money and Finance* 14:1, 3–26.

Longin, F. & B. Solnik (2001). Extreme correlation of international equity markets. *Journal of Finance* 56:2, 649–676.

Longin, F. M. (1996). The asymptotic distribution of extreme stock market returns. *Journal of Business* 69:3, 383–408.

Loretan, M. & W. B. English (2000). Evaluating “correlation breakdowns” during periods of market volatility. International Finance Discussion Paper 658, Board of Governors of the Federal Reserve System, Washington DC.

Loretan, M. & P. C. Phillips (1994). Testing the covariance stationarity of heavy-tailed time series. *Journal of Empirical Finance* 1, 211–248.

Lucas, R. E. (1986). Adaptive behavior and economic theory. *Journal of Business* 59:4, 401–426.

Lui, Y.-H. & D. Mole (1998). The use of fundamental and technical analysis by foreign exchange dealers: Hong Kong evidence. *Journal of International Money and Finance* 17:3, 535–545.

Lux, T. (1995). Herd behaviour, bubbles and crashes. *Economic Journal* 105:431, 881–896.

Lux, T. (1996a). Long-term stochastic dependence in financial prices: Evidence from the German stock market. *Applied Economics Letters* 3, 701–706.

Lux, T. (1996b). The stable Paretian hypothesis and the frequency of large returns: an examination of major German stocks. *Applied Financial Economics* 6, 463–475.

Lux, T. (1998). The socio-economic dynamics of speculative markets: interacting agents, chaos, and the fat tails of return distributions. *Journal of Economic Behavior and Organization* 33, 143–165.

Lux, T. (2001). Power laws and long memory. *Quantitative Finance* 1:6, 560–562.

Lux, T. & M. Ausloos (2002). Market fluctuations I: Scaling, multiscaling, and their possible origins. In Bunde, A., J. Kropp & H. J. Schellnhuber, editors, *The Science of Disasters*, pages 373–409, Berlin. Springer.

Lux, T. & M. Marchesi (1999). Scaling and criticality in a stochastic multi-agent model of a financial market. *Nature* 397, 498–500.

Lux, T. & M. Marchesi (2000). Volatility clustering in financial markets: A microsimulation of interacting agents. *International Journal of Theoretical and Applied Finance* 3:4, 675–702.

Lye, J. N. & V. L. Martin (1991). Modelling and testing stationary exchange rate distributions: The case of the generalized Student's t distribution. mimeo, University of Melbourne.

Ma, S.-K. (1976). *Modern Theory of Critical Phenomena*, volume 46 of *Frontiers in Physics*. Addison-Wesley, Reading, Massachusetts.

MacDonald, R. & M. P. Taylor, editors (1992). *Exchange Rate Economics*, volume 1. Edward Elgar, Aldershot, England.

MacKinlay, A. C. (1995). Multifactor models do not explain deviations from the CAPM. *Journal of Financial Economics* 38, 3–28.

MacKinnon, J. G. (1994). Approximate asymptotic distribution functions for unit-root and cointegration tests. *Journal of Business and Economic Statistics* 12:2, 167–176.

MacKinnon, J. G. (1996). Numerical distribution functions for unit root and cointegration tests. *Journal of Applied Econometrics* 11:6, 601–618.

Madan, D. B., P. P. Carr & E. C. Chang (1998). The variance gamma process and option pricing. *European Finance Review* 2, 79–105.

Madan, D. B. & E. Senata (1990). The Variance Gamma (V.G.) model for share market returns. *Journal of Business* 63:4, 511–524.

Maheu, J. M. (2005). Can GARCH models capture long-range dependence? *Studies in Nonlinear Dynamics and Econometrics* 9:4, Article 1.

Malkiel, B. G. (1979). The capital formation problem in the United States. *Journal of Finance* 34:2, 291–306.

Mandelbrot, B. (1963). The variation of certain speculative prices. *Journal of Business* 36:4, 394–419.

Mandelbrot, B. (1966). Forecasts of future prices, unbiased markets, and "martingale" models. *Journal of Business* 39:1, 242–255.

Mandelbrot, B., A. Fisher & L. Calvet (1997). A multifractal model of asset returns. Discussion Paper 1164, Cowles Foundation for Research in Economics, Yale University.

Mandelbrot, B. & H. M. Taylor (1967). On the distribution of stock price differences. *Operations Research* 15, 1057–1062.

Manski, C. F. & D. McFadden, editors (1981). *Structural Analysis of Discrete Data with Econometric Applications.* MIT Press, Cambridge, Massachusetts.

Mantegna, R. N. & H. E. Stanley (1994). Stochastic process with ultraslow cnvergence to a Gaussian: The truncated Lévy flight. *Physical Review Letters* 73:22, 2946–2949.

Mantegna, R. N. & H. E. Stanley (1995). Scaling behaviour in the dynamics of an economic index. *Nature* 376, 46–49.

Mantegna, R. N. & H. E. Stanley (2000). *An Introduction to Econophysics.* Cambridge University Press, Cambridge.

Markowitz, H. (1959). *Portfolio Seclection: Efficient Diversification of Investments.* John Wiley & Sons, New York.

Martikainen, T. (2000). Efficiency and anomalies in the Finnish stock market. In Keim, D. & W. Ziemba, editors, *Security Market Imperfections in World Wide Equity Markets*, pages 390–415. Cambridge University Press.

Matia, K., Y. Ashkenazy & H. E. Stanley (2003). Multifractal properties of price fluctuations of stocks and commodities. *Europhysics Letters* 61:3, 422–428.

Matteo, T. D., T. Aste & M. M. Dacorogna (2005). Long-term memories of developed and emerging markets: Using the scaling analysis to characterize their stage of development. *Journal of Banking and Finance* 29:4, 827–851.

McDonald, J. B. & W. K. Newey (1988). Partially adaptive estimation of regression models via the generalized t distribution. *Economic Theory* 4, 428–457.

McDonald, J. B. & Y. J. Xu (1995). A generalization of the beta distribution with applications. *Journal of Econometrics* 66, 133–152.

Menkhoff, L. (1998). The noise trading approach—questionaire evidence from foreign exchange. *Journal of International Money and Finance* 17:3, 547–564.

Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics* 3, 125–144.

Mikosch, T. (2003a). Long range dependence effects and ARCH modelling. In Doukhan, P., G. Oppenheim & M. Taqqu, editors, *Theory and Application of Long-Range Dependence*, Boston. Birkhäuser Press.

Mikosch, T. (2003b). Modeling dependence and tails of financial time series. In Finkenstaedt, B. & H. Rootzen, editors, *Extreme Values in Finance, Telecommunications, and the Environment*, pages 185–286. Chapman and Hall.

Mikosch, T. & C. Stărică (2004). Nonstationarities in financial time series, the long-range dependence, and the IGARCH effects. *Review of Economics and Statistics* 86:1, 378–390.

Milgrom, P. & N. Stokey (1982). Information, trade and common knowledge. *Journal of Economic Theory* 26:1, 17–27.

Mills, T. C. (1995). Modelling skewness and kurtosis in the London stock exchange FT-SE index return distributions. *Statistician* 44:3, 323–332.

Mills, T. C. (1997). Stylized facts on the temporal and distributional properties of daily FT-SE returns. *Applied Financial Economics* 7, 599–604.

Minsky, H. P. (1977). The financial instability hypothesis: An interpretation of Keynes and an alternative to 'standard' theory. *Nebraska Journal of Economics and Business* 16, 5–16.

Monroe, I. (1978). Processes that can be embedded in Brownian motion. *Annals of Probability* 6:1, 42–56.

Moors, J. J. A. (1988). A quantile alternative for kurtosis. *Statistician* 37, 25–32.

Morgan, I. G. (1976). Stock prices and heteroscedasticity. *Journal of Business* 49, 496–508.

Müller, U. A., M. M. Dacorogna, R. D. Davé, R. B. Olsen, O. V. Pictet & J. E. von Weizsäcker (1997). Volatilities of different time resolutions – analyzing the dynamics of market components. *Journal of Empirical Finance* 4, 213–239.

Muth, J. F. (1961). Rational expectations and the theory of price movements. *Econometrica* 29:3, 315–335.

Muzy, J. F., J. Delour & E. Bacry (2000). Modelling fluctuations of financial time series: from cascade process to stochastic volatility model. *European Physics Journal B* 17, 537–548.

Nelson, D. B. (1991). Conditional heteroskedasticity in asset returns: A new approach. *Econometrica* 59:2, 347–370.

Nicholson, F. (1968). Price ratios in relation to investment results. *Financial Analysts Journal* 1968:1, 105–109.

Niederhoffer, V. (1971). The analysis of world events and stock prices. *Journal of Business* 44:2, 193–219.

Officer, R. R. (1972). The distribution of stock returns. *Journal of the American Statistical Association* 67:340, 807–812.

Oldfield, G. S. & R. J. Rogalski (1980). A theory of common stock returns over trading and non-trading periods. *Journal of Finance* 35:3, 729–751.

Oldfield, G. S., R. J. Rogalski & R. A. Jarrow (1977). An autoregressive jump process for common stock returns. *Journal of Financial Economics* 5, 389–418.

Osborne, M. F. M. (1959). Brownian motion in the stock market. *Operations Research* 7, 145–173. Reprinted in Cootner (1964), pages 100–128.

Pagan, A. (1996). The econometrics of financial markets. *Journal of Empirical Finance* 3, 15–102.

Pasquini, M. & M. Serva (1999). Multiscale behaviour of volatility autocorrelations in a financial market. *Economics Letters* 65, 275–279.

Peiró, A. (1994). The distribution of stock returns: International evidence. *Applied Financial Economics* 4, 431–439.

Peng, C.-K., S. V. Buldyrev, S. Havlin, M. Simons, H. E. Stanley & A. L. Goldberger (1994). Mosaic organization of dna nucleotides. *Physical Review E* 49:2, 1685–1689.

Perry, P. R. (1983). More evidence on the nature of the distribution of security returns. *Journal of Financial and Quantitative Analysis* 18:2, 211–221.

Pindyck, R. S. (1984). Risk, inflation, and the stock market. *American Economic Review* 74:3, 335–351.

Plerou, V., P. Gopikrishnan, L. A. N. Amaral, M. Meyer & H. E. Stanley (1999). Scaling of the distribution of price fluctuations of individual companies. *Physical Review E* 60, 6519–6529.

Praetz, P. D. (1972). The distribution of share price changes. *Journal of Business* 45:1, 49–55.

Press, S. J. (1967). A compound events model for security prices. *Journal of Business* 40:3, 317–335.

Raberto, M., E. Scalas, G. Cuniberti & M. Riani (1999). Volatility in the italian stock market: An empirical study. *Physica A* 269, 148–155.

Ramsey, J. B. (1996). On the existence of macro variables and of macro relationships. *Journal of Economic Behavior and Organization* 30, 275–299.

Regúlez, M. & A. Zarraga (2002). Common features between stock returns and trading volume. *Applied Financial Economics* 12, 885–893.

Reinganum, M. R. (1983). The anomalous stock market behavior of small firms in January: Empirical tests for tax-loss selling effects. *Journal of Financial Economics* 12:1, 89–104.

Reiss, R.-D. & M. Thomas (1997). *Statistical Analysis of Extreme Values.* Birkhäuser, Berlin.

Richardson, M. & T. Smith (1994). A direct test of the mixture of distributions hypothesis: Measuring the daily flow of information. *Journal of Financial and Quantitative Analysis* 29:1, 101–116.

Robert, C. P. & G. Casella (2004). *Monte Carlo Statistical Methods.* Springer Texts in Statistics. Springer, New York, 2nd edition.

Rogalski, R. J. (1978). The dependence of prices and volume. *Review of Economics and Statistics* 60:2, 268–274.

Roll, R. (1988). R^2. *Journal of Finance* 43:2, 541–566.

Rosenberg, B., K. Reid & R. Lanstein (1985). Persuasive evidence of market inefficiency. *Journal of Portfolio Management* 1985:1, 9–16.

Rouwenhorst, K. G. (1998). International momentum strategies. *Journal of Finance* 53:1, 267–284.

Rozeff, M. S. & W. R. Kinney (1976). Capital market seasonality: The case of stock returns. *Journal of Financial Economics* 3:4, 379–402.

Rubinstein, M. (1975). Securtiy market efficiency in an Arrow-Debreu economy. *American Economic Review* 65:5, 812–824.

Rubinstein, M. (2001). Rational markets: Yes or no? the affirmative case. *Financial Analysts Journal* 57:3, 15–29.

Russel, T. & R. Thaler (1985). The relevance of quasi rationality in competitive markets. *American Economic Review* 75:5, 1071–1082.

Sahimi, M. (1994). *Application of Percolation Theory.* Taylor and Francis, London.

Samuelson, P. A. (1965). Proof that properly anticipated prices fluctuate randomly. *Industrial Management Review* 6:2, 41–49.

Samuelson, P. A. (1973). Proof that properly discounted present values of assets vibrate randomly. *Bell Journal of Economics and Management Science* 4:2, 369–374.

Scheinkman, J. A. & B. LeBaron (1989). Nonlinear dynamics and stock returns. *Journal of Business* 62:3, 311–337.

Schimmler, J. (1973). Speculation, profitability, and price stability – a formal approach. *Review of Economics and Statistics* 55, 110–114.

Schwert, G. W. (1989). Why does stock market volatility change over time? *Journal of Finance* 44:5, 1115–1153.

Schwert, G. W. (1990). Indexes of U.S. stock prices from 1802 to 1987. *Journal of Business* 63:3, 399–426.

Schwert, G. W. (2003). Anomalies and market efficiency. In Constantinides, G., M. Harris & R. Stulz, editors, *Handbook of the Economics of Finance*, pages 937–972, Amsterdam. Elsevier Science B.V.

Sethi, R. (1996). Endogenous regime switching in speculative markets. *Structural Change and Economic Dynamics* 7, 99–118.

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance* 19:3, 425–442.

Shepard, N. (1996). Statistical aspects of ARCH and stochastic volatility. In Cox, D. R., D. V. Hinkley & O. E. Barndorff-Nielsen, editors, *Time Series Models in Econometrics, Finance and other Fields*, pages 1–68, London. Chapman & Hall.

Shleifer, A. & L. H. Summers (1990). The noise trader approach to finance. *Journal of Economic Perspectives* 4:2, 19–33.

Shleifer, A. & R. W. Vishny (1997). The limits of arbitrage. *Journal of Finance* 52:1, 35–55.

Sias, R. (2004). Institutional herding. *Review of Financial Studies* 17:1, 165–206.

Simkowitz, M. A. & W. L. Beedles (1980). Asymmetric stable distributed security returns. *Journal of the American Statistical Association* 75:370, 306–312.

Simon, H. A. (1957). *Models of Man.* Wiley, New York, NY.

Simon, H. A. (1979). Rational decision making in business organizations. *American Economic Review* 69:4, 493–513.

Smith, J. B. (1981). *The Probability Distribution of Market Returns: A Logistic Hypothesis.* PhD thesis, University of Utah.

Smith, V. L., G. L. Suchanek & A. W. Williams (1988). Bubbles, crashes, and endogeneous expectations in experimental spot asset markets. *Econometrica* 56:5, 1119–1151.

Sornette, D., D. Stauffer & H. Takayasu (2002). Market fluctuations II: Multiplicative and percolation models, size effects, and predictions. In Bunde, A., J. Kropp & H. J. Schellnhuber, editors, *The Science of Disasters*, pages 411–435, Berlin. Springer.

Stanley, M. H. R., L. A. N. Amaral, B. S. V., S. Havlin, H. Leschhorn, P. Maass, M. A. Salinger & H. E. Stanley (1996). Can statistical physics contribute to the science of economics? *Fractals* 4:3, 415–425.

Stărică, C. & T. Mikosch (2000). Is it really long memory we see in financial returns? In Embrechts, P., editor, *Extremes and Integrated Risk Management*. Risk Books.

Stărică, C. & O. Pictet (1997). The tales the tails of GARCH processes tell. Available at `http://www.math.chalmers.se/~starica/`.

Stein, E. M. & J. C. Stein (1991). Stock price distributions with stochastic volatility. *Review of Financial Studies* 4:4, 727–752.

Stiglitz, J. E. (1989). Imperfect information in the product market. In Schmalensee, R. & R. Willig, editors, *Handbook of Industrial Organization*, volume 1, pages 771–847, Amsterdam. North-Holland. cited in Kirman (1992).

Sullivan, R., A. Timmermann & H. White (1998). Dangers of data-driven inference: The case of calendar effects in stock returns. UCSD Economics Discussion Paper 98-16, University of California, San Diego.

Sullivan, R., A. Timmermann & H. White (1999). Data-snooping, technical trading rule performance, and the bootstrap. *Journal of Finance* 54:5, 1647–1691.

Sunder, S. (1995). Experimental asset markets: A survey. In Kagel, J. & A. Roth, editors, *Handbook of Experimental Economics*, Princeton. Princeton University Press.

Tauchen, G. E. & M. Pitts (1983). The price variability-volume relationship on speculative markets. *Econometrica* 51:2, 485–506.

Tauchen, G. E., H. Zhang & M. Liu (1996). Volume, volatility, and leverage: A dynamic analysis. *Journal of Econometrics* 74, 177–208.

Taylor, M. P. & H. Allen (1992). The use of technical analysis in the foreign exchange market. *Journal of International Money and Finance* 11, 304–314.

Taylor, S. J. (1986). *Modelling Financial Time Series*. Wiley.

Taylor, S. J. (1994). Modelling stochastic volatility: A review and comparative study. *Mathematical Finance* 4:2, 183–204.

Teichmoeller, J. (1971). A note on the distribution of stock price changes. *Journal of the American Statistical Association* 66:334, 282–284.

Tesfatsion, L. & K. Judd, editors (2006). *Handbook of Computational Economics*, volume 2. North-Holland, Amsterdam.

Thaler, R. H. (1999). The end of behavioral finance. *Financial Analysts Journal* 1999:6, 12–17.

Theodossiou, P. (1998). Financial data and the skewed generalized t distribution. *Mangement Science* 44:12, 1650–1661.

Tucker, A. L. (1992). A reexamination of finite- and infinite-variancee distributions as models for stock returns. *Journal of Business and Economic Statistics* 10:1, 73–81.

Tukey, J. W. (1977). *Exploratory Data Analysis.* Addison-Wesley, Reading, Massachusetts.

Tversky, A. & D. Kahneman (1974). Judgement under uncertainty: Heuristics and biases. *Science* 185:4157, 1124–1131.

Tversky, A. & D. Kahneman (1981). The framing of decisions and the psychology of choice. *Science* 211:4481, 453–458.

Upton, D. E. & D. S. Shannon (1979). The stable Paretian distribution, subordinated stochastic processes, and asymptotic lognormality: An empirical investigation. *Journal of Finance* 34:4, 1031–1039.

Vassilicos, J., A. Demos & F. Tata (1993). No evidence of chaos but some evidence of multifractals in the foreign exchange and the stocks markets. In Crily, A., E. R.A. & H. Jones, editors, *Applications of Fractals and Chaos*, pages 249–265, Berlin. Springer.

Veneziano, D., G. E. Moglen & R. L. Bras (1995). Multifractal analysis: Pitfalls of standard procedures and alternatives. *Physical Review E* 52:2, 1387–1398.

Walter, A. & F. M. Weber (2006). Herding in the German mutual fund industry. *European Financial Management* 12:3, 375–406.

Weidlich, W. (1991). Physics and the social science—the approach of synergetics. *Physics Reports* 204:1, 1–163.

Weidlich, W. (2002). *Sociodynamics.* Taylor & Francis, London.

Weidlich, W. & G. Haag (1983). *Concepts and Models of a Quantitative Sociology*, volume 14 of *Springer Series in Synergetics*. Springer, Berlin.

Westerfield, R. (1977). The distribution of common stock price changes. *Journal of Financial and Quantitative Analysis* 12, 743–765.

Westerhoff, F. H. (2004). Multiasset market dynamics. *Macroeconomic Dynamics* 8, 596–616.

Wichern, D. W., R. B. Miller & D.-A. Hsu (1976). Changes of variance in first-order autoregressive time series models-with an application. *Applied Statistics* 25:3, 248–256.

Womack, K. L. (1996). Do brokerage analysts' recommendations have investment value? *Journal of Finance* 51:1, 137–167.

Working, H. (1934). A random-difference series for use in the analysis of time series. *Journal of the American Statistical Association* 29, 11–24.

Ying, C. C. (1966). Stock market prices and volumes of sales. *Econometrica* 34:3, 676–685.

Zeeman, E. C. (1974). On the unstable behaviour of stock exchanges. *Journal of Mathematical Economics* 1, 39–49.

Ziemba, W. T. (1991). Japanese security market regularities: Monthly, turn of the month and year, holiday and golden week effects. *Japan and the World Economy* 3, 119–146.

Ziemba, W. T. (1994). World wide security market regularities. *European Journal of Operational Research* 74, 198–229.

Zumbach, G. (2004). Volatility processes and volatility forecast with long memory. *Quantitative Finance* 4:1, 70–86.

A Appendix

A1 Matlab code for replication of the simulation study by Lux & Marchesi

```
function returns = LM2000(T)
% LM2000 replicates Lux/Marchesi (2000) IJTAF 3(4), 675-702
% SYNTAX:   RETURNS = LM2000(T)
% INPUT:    T = scalar number of return observations to simulate
% OUTPUT:   RETURNS = (T*1) simulated logreturns
%
% NOTE:     Choice of parameter set by uncommenting (that is,
%           removing the leading % in front of) the relevant
%           block titled 'Parameter set I' to 'Parameter set IV'.
%
% written by Bernd Pape, University of Vaasa, Finland

tic;        %start clock

% initialize random variables (Statistics Toolbox User Guide p. 2-11)
state = 137;
rand('state', state);
randn('state', state);


%   Technical parameters

steps = 500;    %number of microsteps per integer time step
plag = 100;     %number of microsteps in determination of pdot


%   Constant model parameters:

N = 500;    %number of agents
nmin = 4;   %minimum number of agents in each strategy
pf = 10;    %fundamental price
r = 0.004;  %nominal dividends of the asset
R = 0.0004; %ecomonies' average rate of return


%   Parameter set I:

v1 = 3;    %integer time frequency of optimist/pessimist revaluation
v2= 2;     %integer time frequency of chartist/fundament. revaluation
```

```
beta = 6;  %reaction speed of the auctioneer in integer time steps
tc = 10/N;  %trading volume of average chartist
tf = 5/N;   %trading volume of avg. fund. on 1 currency unit mispricing
a1 = 0.6;   %importance of opinion index x for chartist expectation
a2 = 0.2;   %importance of price change pdot for chartist expectation
a3 = 0.5;   %importance of profit differentials for c/f switches
s = 0.75;   %discount factor
sigma = 0.05;    %imprecision in excess demand perception
% %   Parameter set II:
%
% v1 = 4;    %integer time frequency of optimist/pessimist revaluation
% v2= 1;     %integer time frequency of chartist/fundament. revaluation
% beta = 4;  %reaction speed of the auctioneer in integer time steps
% tc = 7.5/N;  %trading volume of average chartist
% tf = 5/N;   %trading volume of avg. fund. on 1 currency unit mispricing
% a1 = 0.9;   %importance of opinion index x for chartist expectation
% a2 = 0.25;   %importance of price change pdot for chartist expectation
% a3 = 1;   %importance of profit differentials for c/f switches
% s = 0.75;   %discount factor
% sigma = 0.1;    %imprecision in excess demand perception
% %   Parameter set III:
%
% v1 = 0.5;    %integer time frequency of optimist/pessimist revaluation
% v2= 0.5;     %integer time frequency of chartist/fundament. revaluation
% beta = 2;  %reaction speed of the auctioneer in integer time steps
% tc = 10/N;  %trading volume of average chartist
% tf = 10/N;   %trading volume of avg. fund. on 1 currency unit mispricing
% a1 = 0.75;   %importance of opinion index x for chartist expectation
% a2 = 0.25;   %importance of price change pdot for chartist expectation
% a3 = 0.75;   %importance of profit differentials for c/f switches
% s = 0.8;   %discount factor
% sigma = 0.1;    %imprecision in excess demand perception
% %   Parameter set IV:
%
% v1 = 2;    %integer time frequency of optimist/pessimist revaluation
% v2= 0.6;     %integer time frequency of chartist/fundament. revaluation
% beta = 4;  %reaction speed of the auctioneer in integer time steps
% tc = 5/N;  %trading volume of average chartist
% tf = 5/N;   %trading volume of avg. fund. on 1 currency unit mispricing
% a1 = 0.8;   %importance of opinion index x for chartist expectation
% a2 = 0.2;   %importance of price change pdot for chartist expectation
```

```
% a3 = 1;   %importance of profit differentials for c/f switches
% s = 0.75;   %discount factor
% sigma = 0.05;    %imprecision in excess demand perception



%   Initialization of aggregated trading volume

Tc = N*tc;  %aggregated chartists trading volume
Tf = N*tf;  %aggreg. fundamentalists trading volume per unit mispricing


%   Initialization of vector and matrix dimensions

precent = repmat(pf,plag,1); %(plag*1) vector of recent prices
phist = zeros(T,1); %(T*1) vector of price history at integer time steps
xhist = zeros(T,1); %(T*1) vector of opinion index history
zhist = zeros(T,1); %(T*1) history of fraction of chartists

hchist = zeros(T,1);    %(T*1) history of chartist holdings
hfhist = zeros(T,1);    %(T*1) history of fundamentalist holdings
cchist = zeros(T,1);    %(T*1) history of chartist cash
cfhist = zeros(T,1);    %(T*1) history of fundamentalist cash


expU = zeros(3,2);  %(3*2) potential exponentials and their inverses


%   Upper threshold for fraction of chartists for stationary dynamics
%   from solving quadratic equation (condition 1, page 686) for z

a = 2*beta*Tc*(a2-a3/pf);                    %quadratic term in z
b = 2*v1*(a1-1)+beta*(2*a3*Tc/pf+Tf);       %linear term in z
zmax = sqrt((b/(2*a))^2+beta*Tf/a)-b/(2*a); %upper crossing with 0


%   Randomly initialize population of trading strategies
%   with at least nmin agents in every stratgy

% number of chartists is random number between 0 and zmax*N,
% but not less than 2*nmin and not more than N-nmin
nc = min([max([fix(rand*zmax*N) 2*nmin]) N-nmin]);
nf = N-nc;               %number of fundamentalists
% number of optimists is random number between 0 and nc,
% but not less than nmin and not more than nc-nmin
np = min([max([round(rand*nc) nmin]) nc-nmin]);
nm = nc-np;              %number of pessimistic chartists
```

```
%   Initialization of state variables

p = pf;          %(scalar) trading price initialized at fundamental price
x = (np-nm)/nc; %(scalar) opinion index [-1,1]
z = nc/N;        %(scalar) chartist index [0,zmax], zmax<1
pdot = 0;        %(scalar) approx. derivative of trading price wrt. time

chold = 0;       %(scalar) stocks owned by representative chartist
fhold = 0;       %(scalar) stocks owned by repr. fundamentalist
ccash = 0;       %(scalar) cash owned by representative chartist
fcash = 0;       %(scalar) cash owned by repr. fundamentalist


%   Simulation loop with (p,x,z) recording at integer time steps

for t = 1:T     %start outer loop over integer time steps

    for st = 1:steps %start inner loop over micro time steps

        % Calculation of scalar strategy changing potentials
        U1 = a1*x+a2*pdot/v1;       %optimist/pessimist potential (p.682)
        oprofit = (r+pdot/v2)/p-R;  %optimist profit = pessimist loss
        fprofit = s*abs((pf-p)/p);  %fundamentalists profit (p.683)
        U21 = a3*(oprofit-fprofit); %optimist/fundam. potential (p.683)
        U22 = -a3*(oprofit+fprofit);%pessim./fundam. potential (p.683)

        % (3*2) exponentials of potentials and of negative potentials
        expU(:,1) = exp([U1; U21; U22]);    %exponentials of potentials
        expU(:,2) = ones(3,1)./expU(:,1);   %their inverses = exp(-U)

        % Calculation of (1*2)/(2*2) population transition probabilities
        ppm = v1*nc/N*expU(1,:)/steps;  %(1*2) [pi_+-, pi_-+] (p.682)
        pcf = v2/N*[np nf; nm nf].*expU(2:3,:)/steps; %[+f,f+;-f,f-](683)

        %(1*2) binomial draws of agents leaving their strategy
        npout = [fastbin(np, ppm(2)) fastbin(np, pcf(1,2))];  %[-+,f+]
        nmout = [fastbin(nm, ppm(1)) fastbin(nm, pcf(2,2))];  %[+-,f-]
        nfout = [fastbin(nf, pcf(1,1)) fastbin(nf, pcf(2,1))];%[+f,-f]

        % Do not allow less than nmin agents in any strategy
        if any([np-sum(npout) nm-sum(nmout) nf-sum(nfout)]<nmin)
            while np-sum(npout)<nmin
                npout = max([npout-ones(1,2);zeros(1,2)]);
            end
            while nm-sum(nmout)<nmin
```

```
                nmout = max([nmout-ones(1,2);zeros(1,2)]);
            end
            while nf-sum(nfout)<nmin
                nfout = max([nfout-ones(1,2);zeros(1,2)]);
            end
        end

        % Update of the strategy populations
        np = np+nmout(1)+nfout(1)-sum(npout);   %inflow m,f - outflow
        nm = nm+npout(1)+nfout(2)-sum(nmout);   %inflow p,f - outflow
        nf = nf+npout(2)+nmout(2)-sum(nfout);   %inflow p,m - outflow
        nc = np + nm;   % number of chartists = optimists + pessimists

        % Keep fraction of chartists below zmax
        zrel = nc/(N*zmax); %chartist index z as a fraction of zmax
        if zrel > 1         %if more chartists than allowed
            np = fix(np/zrel);  %reduce number of optimists
            nm = fix(nm/zrel);  %reduce number of pessimists
            nc = np + nm;       %update number of chartists
            nf = N - nf;        %update number of fundamentalists
        end

        % Calculation of excess demand (Lux p.684)
        EDc = (np-nm)*tc;       %excess demand by chartists
        EDf = nf*tf*(pf-p);     %excess demand by fundamentalists
        ED = EDc + EDf;         %overall excess demand

        % Calculation of new aggregate holdings
        chold = chold + EDc; %new aggregate chartist holdings
        fhold = fhold + EDf; %new aggregate fundamentalist holdings

        % Update of the trading price p
        ppadj = beta*(ED+sigma*randn)*100/steps;%price adjust. probab.
        p=p+sign(ppadj)*(abs(ppadj)>rand)/100;  %adjustment unit 1 cent

        % Update of aggregate traders cash
        ccash = ccash - EDc*p;   %new chartists aggregate chash
        fcash = fcash - EDf*p;   %new fundamentalists aggr. cash

        % Update of pdot and precent
        pdot = steps*(p-precent(1))/plag;   %=dp/dt, dt=plag/steps
        precent(1:plag-1)=precent(2:plag);  %move recent price history
        precent(plag) = p;                  %insert current price

        % Update of the remaning state variables x and z
        x = (np-nm)/nc;                     %opinion index
        z = nc/N;                           %chartist index
```

```
    end % end of inner loop over micro time steps

    % Upadate history of state variables at integer time steps
    phist(t) = p;   %(T*1) trading price history
    xhist(t) = x;   %(T*1) opinion index history
    zhist(t) = z;   %(T*1) chartist index history

    % Upate history of traders' holdings and cash
    hchist(t) = chold;  %(T*1) chartist holdings history
    hfhist(t) = fhold;  %(T*1) fundamentalist holdings history
    cchist(t) = ccash;  %(T*1) chartist cash history
    cfhist(t) = fcash;  %(T*1) fundamentalist cash history


end % end of outer loop over integer time steps


%   Claculate logreturns

lnp = log([pf; phist]);          %((T+1)*1) vector of logarithmic prices
rhist = lnp(2:T+1)-lnp(1:T);     %(T*1) vector of logreturns


%   Calculate history of traders wealth

cwealth = hchist.*phist + cchist;   %(T*1) chartist wealth
fwealth = hfhist.*phist + cfhist;   %(T*1) fundamentalist wealth


%   Plot history of Lux' state variables

figure;
subplot(4,1,1), plot(phist), title('Trading Price');
subplot(4,1,2), plot(rhist), title('Logreturns');
subplot(4,1,3), plot(zhist); title('Chartist Index');
subplot(4,1,4), plot(xhist); title('Opinion Index');


%   Plot history of traders holdings and wealth

figure;
subplot(2,1,1), plot([hchist,hfhist]), title('Traders holdings');
subplot(2,1,2), plot([cwealth,fwealth]), title('Traders wealth');


%   Plot history of marketmaker holdings and wealth
```

```
figure;
subplot(2,1,1), plot(-(hchist+hfhist)), title('Market maker holdings');
subplot(2,1,2), plot(-(cwealth+fwealth)), title('Market maker wealth');


%   Produce function output

returns = rhist;    %(T*1) simulated logreturns

toc %end clock


function k = fastbin(n,p)
%FASTBIN generates binomial random variates optimized for speed
%K = FASTBIN(N,P) generates a random number from the
%binomial distribution with sample size N and probability P.
%Uses the BINV algorithm described in:
%Voratas Kachitvichyanukul and Bruce W. Schmeiser (1988):
%Binomial Random Variate Generation
%Communications of the ACM 31, 216-222
%
% written by Bernd Pape

    %BINV algorithm starts here
    %Step 1
    q = 1-p; s = p/q; a = (n+1)*s; r = q^n;

    %Step 2
    u = rand; k = 0;

    %Step 3 + 4
    while u > r
        u = u - r;
        k = k + 1;
        r = ((a/k)-s)*r;
    end
    %end of BINV algorithm
```

A2 Matlab code for tail index estimation

```
function [k,summary,alpha] = tailtest(ret,nobs)
% TAILTEST Hill estimates of tail index and kurtosis
%-------------------------------------------------------------
%   USAGE: [K,SUMMARY,ALPHA] = TAILTEST(RET,NOBS) performs
%   LENGTH(RET)/NOBS Hill estimates of the tail index upon NOBS
%   observations of the abolute value of the return series RET.
%   The length of RET must be an integer multiple of NOBS.
%   The default value of NOBS is 2000.
%-------------------------------------------------------------
%   OUTPUT:
%   K is the kurtosis estimate for the full return series RET.
%   ALPHA is a structure of Hill estimates for the tail index:
%   ALPHA.P025: tail index based upon upper 2.5% tail of NOBS
%   ALPHA.P05:  tail index based upon upper 5% tail of NOBS
%   ALPHA.P10:  tail index based upon upper 10% tail of NOBS
%   All fields of ALPHA are ((LENGTH(RET)/NOBS)*1) vectors.
%   SUMMARY is a (3*4) matrix containing the minimum, median
%   and maximum tail estimate for each of the three thresholds.
%-------------------------------------------------------------
%   REFERENCE:
%   Buce M. Hill: A simple general appraoch to inference
%   about the tail of a distribution (1975),
%   Annals of Statistics 3(5), 1163-1174
%
%   NOTE: Uses an algorithm taken from the command HILLPLOT
%   of the open source EVIM software package, developed by
%   Ramazan Gencay, Faruk Selcuk and Aburrahman Uluglyagci,
%   available from http://www.bilkent.edu.tr/~faruk.
%
%   written by Bernd Pape

% Check number of input arguments

if nargin == 0 or nargin > 2
    error('Wrong number of input arguments to TAILTEST.')
elseif nargin == 1
    nobs = 2000;     %default number of observations per test
end


% Check whether length of RET is an integer multiple of NOBS

if mod(length(ret),nobs)
    error('The length of RET must be an integer multiple of NOBS.')
```

```
end


% Initialize output variables

tests = length(ret)/nobs;        %# estimates to be produced
alpha.p025 = zeros(tests,1);     %tail index with 2.5% threshold
alpha.p05 = zeros(tests,1);      %tail index with 5% threshold
alpha.p10 = zeros(tests,1);      %tail index with 10% threshold


% Start loop over number of Hill estimates to be produced

for test = 1:tests

    % Select absolute return window from return series
    data = abs(ret((test-1)*nobs+1:test*nobs)); %(NOBS*1)

    % Calculate tail index estimates (see code in HILLPLOT)
    ordered = flipud(sort(data));   %(NOBS*1) upper order stat.
    ordered = ordered(ordered>0);   %restrict to strictly pos.
    n = length(ordered);            %length of restricted data
    loggs = log(ordered);           %(N*1) logarithms of above
    avesumlog = cumsum(loggs)./(1:n)';   %(N*1) avg.logs
    diffs = avesumlog-loggs;        %(N*1) xi-estimates
    diffs = [NaN; diffs(2:n)];      %replace 0 with NaN
    hill = 1./diffs;                %(N*1) Hill estimates

    % Read out relevant thresholds
    alpha.p025(test) = hill(floor(0.025*n)); %2.5% thresh.
    alpha.p05(test) = hill(floor(0.05*n));   %5% thresh.
    alpha.p10(test) = hill(floor(0.1*n));    %10% thresh.

end %End of loop over number of Hill estimates to be produced

k = kurtosis(ret);  %Sample kurtosis of full return series

% Generate summary matrix
alphamat = [alpha.p025 alpha.p05 alpha.p10];    %(tests*3)
minalpha = min(alphamat,[],1);  %(1*3) smallest alpha found
medalpha = median(alphamat,1);  %(1*3) median alpha found
maxalpha = max(alphamat,[],1);  %(1*3) largest alpha found
summary = [[0.025; 0.05; 0.1] minalpha' medalpha' maxalpha'];
```

A3 Matlab code for log-periodogram regression

```
function[summary,rejections,d] = dtest(y,nobs,ci)
% DTEST log-periodogram regression (Geweke/Porter-Hudak)
%----------------------------------------------------------------
%   USAGE: [SUMMARY,REJECTIONS,D] = DTEST(Y,NOBS,CI) performs
%   LENGTH(Y)/NOBS log-periodogram regressions to determine the
%   fractional differencing parameter d of the time series Y.
%   The length of Y must be an integer multiple of NOBS.
%   The default value of NOBS is 2000.
%
%   DTEST performs also LENGTH(Y)/NOBS two-sided tests of the
%   null hypothesis H0: d=0 with confidence interval CI.
%   The default value of CI is 0.9, equivalent to two
%   one-sided tests with rejection probability 5%.
%----------------------------------------------------------------
%   OUTPUT:
%   SUMMARY (1*3):  lowest, median, and highest estimate of d
%   REJECTIONS(1*2):percentage of rejections indicating d<0, d>0
%   D(LENGTH(Y)/NOBS): fractional differencing parameter esimates
%----------------------------------------------------------------
%   REFERENCES:
%   John Geweke and Susan Porter-Hudak: The Estimation
%   and Application of Long Memory Time Series (1983),
%   Journal of Time Series Analysis 4 (4) pp. 221-238;
%   Thomas Lux: Long-term dependence in financial prices,
%   evidence from the German stock market (1996),
%   Applied Economic Letters 3 pp. 701-706;
%   James D. Hamilton: Time Series Analysis (1994) p.158
%
% NOTE: calls the functions ACF and OLS from the open source
% ECONOMETRICS TOOLBOX by James P. LeSage available from
% http://www.econ.utoledo.edu.
%
% written by Bernd Pape

% Check number of input arguments

if nargin == 0 or nargin > 3
    error('Wrong number of input arguments to DTEST.')
elseif nargin  < 3
    ci = 0.9;       %default confidence interval is 90%
    if nargin < 2
        nobs = 2000;%default number of observations per test
    end
end
```

```
% Check whether length of Y is integer multiple of NOBS

if mod(length(y),nobs)
    error('The length of Y must be an integer multiple of NOBS.')
end


% Initialize output variables

tests = length(y)/nobs;     %# estimates to be produced
d = zeros(tests,1);         %(test*1) estimates for d
rejections = zeros(1,2);    %(1*2) % rejections of H0 vs d<0, d>0


% Generate independent variables for log-periodogramm regressions

m = floor(sqrt(nobs));      %# lowest Fourier freq. considered
j = (1:m)';                 %(m*1) Fourier frequency indexes
lambda = 2*pi*j/nobs;       %(m*1) Fourier frequencies
ind = [ones(m,1) log(4*sin(lambda/2).^2)];   %(m*2) indep. var.


% Pre-calculate cosine factors for calculation of periodogram

cosine = cos(lambda*(1:nobs-1));    %(m*(nobs-1)) Hamilton p.158


% Start loop over # log-periodogram regressions to be performed

for test = 1:tests

    %Calculate periodogram of sub-sample
    data = y((test-1)*nobs+1 : test*nobs);  %(nobs*1) ts window
    acfproc = acf(data, nobs-1);    %structure from calling acf
    rho = acfproc.ac;   %((nobs-1)*1) autocorrelation coeff.'s
    I = var(data)/(2*pi)*(1+2*cosine*rho);  %(m*1) periodogram

    %Perform log-periodogram regression
    dep = log(I);                       %(m*1) dependent variable
    regression = ols(dep,ind);          %structure from calling ols
    d(test) = -regression.beta(2);  %(scalar) slope coefficient

end %end of loop over # log-periodogram regressions
```

```
% Calculate # rejections of H0: d=0 using Lux (1996) p.704

sqdev = ind(:,2)-sum(ind(:,2))/m;       %(m*1) squared deviations
sigma = pi/sqrt(sqdev'*sqdev);          %asympt. stdev. of d est.
critd = norminv([0.5-ci/2 0.5+ci/2],0,sigma); %critical values
rejections(1) = sum(d<critd(1))/tests;  %rejections favouring d<0
rejections(2) = sum(d>critd(2))/tests;  %rejections favouring d>0

% Calculate summary statistics

summary = [min(d) median(d) max(d)];    %lowest, median, higest d
```

A4 Matlab code for Dickey-Fuller test including a constant

```
function [rho,taustat,pctile_c,tests] = df1976c(y)
% DF1976C Dickey Fuller test including a constant
%---------------------------------------------------------------
%   USAGE: [RHO,TAUSTAT,PCTILE_C,TESTS] = DF1976C(Y)
%   performs TESTS=LENGTH(Y)/500 Dickey Fuller tests including
%   a constant upon 500 observations of the time series Y
%   using the critical values from Fuller (1976).
%   The length of Y must be an integer multiple of 500.
%   NOTE: It's also possible to use noninteger multiples of 500
%   by corresponding modification of the first line of
%   executable code: nobs = (new # observations in each test),
%   but the critical values for 500 observations will still
%   be used, which differ only in the last digit from inf. obs.
%---------------------------------------------------------------
%   OUTPUT:
%   RHO is a ((LENGTH(Y)/500)*1) vector of estimated values
%   for rho in the regression y_t = c + rho*y_{t-1} + e_t.
%   TAUSTAT is a ((LENGTH(Y)/500)*1) vector of DF-statistics.
%   PCTILE_C is a structure containing the number of tests
%   resulting in the following percentile of the asymptotic
%   distribution of RHO under the null hypothesis rho = 1
%   using the second panel of table 8.5.2 in Fuller (1976)
%   PCTILE_C.LT025:   P(RHO) < 0.025
%   PCTILE_C.LT05:    0.025 <= P(RHO) < 0.05
%   PCTILE_C.H0:      0.05 <= P(RHO) <= 0.95
%   PCTILE_C.GT95:    0.95 < P(RHO) <= 0.975
%   PCTILE_C.GT975:   0.975 < P(RHO)
%---------------------------------------------------------------
%   REFERENCE:
%   Wayne A. Fuller (1976): Introduction to statistical time series
%   Table 8.5.2 (p.373) first/second panel = with/without constant
%
% NOTE: calls the function OLS from the open source
% ECONOMETRICS TOOLBOX by James P. LeSage available from
% http://www.econ.utoledo.edu.
%
% written by Bernd Pape



nobs = 500; %Fuller has only 25,50,100,250,500,inf observations

```

```
%% Check whether length of Y is integer multiple of NOBS

if mod(length(y),nobs)
    error('The length of Y must be an integer multiple of 500.')
end


% Initialize output variables

tests = length(y)/nobs; %number of tests to be performed
rho = zeros(tests,1);   %(tests*1) estimated AR(1) coefficients
taustat = zeros(tests,1);   %(tests*1) t-statistics of rho


% Start loop over number of DF tests to be performed

for test = 1:tests

    % Generate input for regression
    if test == 1            % NOBS-1 observations in first test
        x = y(1:nobs-1);    %((nobs-1)*1) lagged time series
        x = [x, ones(nobs-1,1)];    %include constant
        dy = y(2:nobs)-x(:,1);   %((nobs-1)*1) change in time series
    else                    % NOBS observations otherwise
        x = y((test-1)*nobs : test*nobs-1);     %lagged ts
        x = [x, ones(nobs,1)];      %include constant
        dy = y((test-1)*nobs+1 : test*nobs)-x(:,1);  %change in ts
    end

    % Perform regression of change in ts upon lagged ts
    regression = ols(dy,x);             %perform regression
    rho(test) = regression.beta(1) + 1; %estimated AR(1) coeff.
    taustat(test) = regression.tstat(1);%t-statistics of y_{t-1}


end %end of loop over number of DF tests to be performed


% (TESTS*1) indicator vectors for percentile distribution
% with constant from second panel in Table 8.5.2 (p.373)

ltc025 = (taustat < -3.13);    %true if pval < 0.025 (inf: -3.12)
ltc05 = (taustat < -2.87);     %true if pval < 0.05 (inf: -2.86)
ltc95 = (taustat <= -0.07);    %true if pval <= 0.95 (inf: -0.07)
ltc975 = (taustat <= 0.24);    %true if pval <= 0.975 (inf: 0.23)
```

```
% create PCTILE_C output structure (correct table: includes constant)

pctile_c.lt025 = sum(ltc025);          %P(RHO) < 0.025
pctile_c.lt05 = sum(ltc05-ltc025);    %0.025 <= P(RHO) < 0.05
pctile_c.H0 = sum(ltc95-ltc05);       %0.05 <= P(RHO) <= 0.95
pctile_c.gt95 = sum(ltc975-ltc95);    %0.95 < P(RHO) <= 0.975
pctile_c.gt975 = tests-sum(ltc975);    %0.975 < P(RHO)


% Print range of RHO to command window
sprintf('The range of RHO is %1.6f to %1.6f.',min(rho),max(rho))
```

A5 Matlab code for error detection in unit root tests

```
function [rho,taustat,pctile_nc,pctile_c,tests] = df1976(y)
% DF1976 Dickey Fuller test without constant or trend
%---------------------------------------------------------------
%   USAGE: [RHO,TAUSTAT,PCTILE_NC,PCTILE_C,TESTS] = DF1976(Y)
%   performs TESTS=LENGTH(Y)/500 Dickey Fuller tests without
%   a constant upon 500 observations of the time series Y
%   using the critical values from Fuller (1976).
%   The length of Y must be an integer multiple of 500.
%---------------------------------------------------------------
%   OUTPUT:
%   RHO is a ((LENGTH(Y)/500)*1) vector of estimated values
%   for rho in the regression y_t = rho*y_{t-1} + e_t.
%   TAUSTAT is a ((LENGTH(Y)/500)*1) vector of DF-statistics.
%   PCTILE_NC is a structure containing the number of tests
%   resulting in the following percentile of the asymptotic
%   distribution of RHO under the null hypothesis rho = 1
%   using the correct table without constant in Fuller (1976):
%   PCTILE_NC.LT025:   P(RHO) < 0.025
%   PCTILE_NC.LT05:    0.025 <= P(RHO) < 0.05
%   PCTILE_NC.H0:      0.05 <= P(RHO) <= 0.95
%   PCTILE_NC.GT95:    0.95 < P(RHO) <= 0.975
%   PCTILE_NC.GT975:   0.975 < P(RHO)
%   PCTILE_C is a structure containing the number of tests
%   resulting in the following percentile of the asymptotic
%   distribution of RHO under the null hypothesis rho = 1
%   using the wrong(!) table with constant in Fuller (1976):
%   PCTILE_C.LT025:   P(RHO) < 0.025
%   PCTILE_C.LT05:    0.025 <= P(RHO) < 0.05
%   PCTILE_C.H0:      0.05 <= P(RHO) <= 0.95
%   PCTILE_C.GT95:    0.95 < P(RHO) <= 0.975
%   PCTILE_C.GT975:   0.975 < P(RHO)
%---------------------------------------------------------------
%   REFERENCE:
%   Wayne A. Fuller (1976): Introduction to statistical time series
%   Table 8.5.2 (p.373) first/second panel = with/without constant
%
% NOTE: calls the function OLS from the open source
% ECONOMETRICS TOOLBOX by James P. LeSage available from
% http://www.econ.utoledo.edu.
%
% written by Bernd Pape

```

```
nobs = 500; %Fuller has only 25,50,100,250,500,inf observations

% Check whether length of Y is integer multiple of NOBS

if mod(length(y),nobs)
    error('The length of Y must be an integer multiple of 500.')
end


% Initialize output variables

tests = length(y)/nobs; %number of tests to be performed
rho = zeros(tests,1);   %(tests*1) estimated AR(1) coefficients
taustat = zeros(tests,1);   %(tests*1) t-statistics of rho


% Start loop over number of DF tests to be performed

for test = 1:tests

    % Generate input for regression
    if test == 1             % NOBS-1 observations in first test
        x = y(1:nobs-1);    %((nobs-1)*1) lagged time series
        dy = y(2:nobs)-x;   %((nobs-1)*1) change in time series
    else                     % NOBS observations otherwise
        x = y((test-1)*nobs : test*nobs-1);     %lagged ts
        dy = y((test-1)*nobs+1 : test*nobs)-x;  %change in ts
    end

    % Perform regression of change in ts upon lagged ts
    regression = ols(dy,x);             %perform regression
    rho(test) = regression.beta + 1;    %estimated AR(1) coeff.
    taustat(test) = regression.tstat;   %t-statistics of y_{t-1}


end %end of loop over number of DF tests to be performed


% (TESTS*1) indicator vectors for percentile distribution
% without constant from first panel in Table 8.5.2 (p.373)

ltnc025 = (taustat < -2.23);    %logical(1) if pval < 0.025
ltnc05 = (taustat < -1.95);     %logical(1) if pval < 0.05
ltnc95 = (taustat <= 1.28);     %logical(1) if pval <= 0.95
```

```
ltnc975 = (taustat <= 1.62);    %logical(1) if pval <= 0.975


% create PCTILE_NC output structure (correct table: no constant)

pctile_nc.lt025 = sum(ltnc025);         %P(RHO) < 0.025
pctile_nc.lt05 = sum(ltnc05-ltnc025);   %0.025 <= P(RHO) < 0.05
pctile_nc.H0 = sum(ltnc95-ltnc05);      %0.05 <= P(RHO) <= 0.95
pctile_nc.gt95 = sum(ltnc975-ltnc95);   %0.95 < P(RHO) <= 0.975
pctile_nc.gt975 = tests-sum(ltnc975);   %0.975 < P(RHO)


% (TESTS*1) indicator vectors for percentile distribution
% with constant from second panel in Table 8.5.2 (p.373)

ltc025 = (taustat < -3.13);    %logical(1) if pval < 0.025
ltc05 = (taustat < -2.87);     %logical(1) if pval < 0.05
ltc95 = (taustat <= -0.07);    %logical(1) if pval <= 0.95
ltc975 = (taustat <= 0.24);    %logical(1) if pval <= 0.975


% create PCTILE_C output structure (wrong table: includes constant)

pctile_c.lt025 = sum(ltc025);         %P(RHO) < 0.025
pctile_c.lt05 = sum(ltc05-ltc025);   %0.025 <= P(RHO) < 0.05
pctile_c.H0 = sum(ltc95-ltc05);      %0.05 <= P(RHO) <= 0.95
pctile_c.gt95 = sum(ltc975-ltc95);   %0.95 < P(RHO) <= 0.975
pctile_c.gt975 = tests-sum(ltc975);   %0.975 < P(RHO)


% Print range of RHO to command window
sprintf('The range of RHO is %1.6f to %1.6f.',min(rho),max(rho))
```

A6 Matlab code for simulation of the allocation model in chapter 5

```
function [stock1,stock2,ewindex,cwindex] = Allocation(T)
% ALLOCATION simulates Asset Allocation model
% SYNTAX: [STOCK1,STOCK2,EWINDEX,CWINDEX] = ALLOCATION(T)
% INPUT:    T = scalar number of observations to simulate
% OUTPUT:   STOCK1 = (T*1) logreturns of stock 1
%           STOCK2 = (T*1) logreturns of stock 2
%           EWINDEX = (T*1) logreturns of equal weighted index
%           CWINDEX = (T*1) logreturns of cap. weighted index
%
% written by Bernd Pape, University of Vaasa, Finland


tic %start clock

% Technical Parameters

steps = 500;   %nuber of microsteps per integer time step
N = 500;       %number of traders

% Model Parameters

pf=zeros(1,2);  %(1*2) logarithmic fundamental prices
v=0.001;        %# stock/strategy revaluations per macro time step
vb=0.04;        %# asset allocation revaluations p. macro time step
a=0.1;          %strength of infection within stocks
ab=0.4;         %strength of infection between stocks / bonds
l=0.5;          %leverage parameter (tc/tf)
s=0.8;          %discount factor for fundamentalist profits


% Initialization of vector and matrix dimensions

p=pf;               %(1*2) vector of current trading prices
dp=zeros(1,2);      %(1*2) price increments at micro time step
phist=zeros(T,2);   %(T*2) logprice history at integer steps
Phist=zeros(T,2);   %(T*2) trading price history at integer steps

c = zeros(1,2);     %(1*2) current chartist utilities

nc=zeros(1,2);      %(1*2) vector of current chartist populations
dnc=zeros(1,2);     %(1*2) increments in chart. pop's at microstep
nchist=zeros(T,2);  %(T*2) history of chartist populations

```

```
nf=zeros(1,2);        %(1*2) vector of current fundamentalist populations
dnf=zeros(1,2);       %(1*2) increments in fundam. pop's at microstep
nfhist=zeros(T,2);    %(T*2) history of fundamentalist populations

pmat=zeros(6,2);      %(6*2) strategy transition prob's in stocks
dnmat=zeros(6,2);     %(6*2) stockinvestors changing strategy
%Entries in the matrices above:
%                               c1->c2    c2->c1
%                               f1->f2    f2->f1
%                               c1->f1    f1->c1
%                               c2->f2    f2->c2
%                               c1->f2    f2->c1
%                               c2->f1    f1->c2

pbmat = zeros(4,2); %(4*2) transitions between bonds and stocks
dnbmat= zeros(4,2); %(4*2) traders switching betw. bonds/ stocks
%Entries in the matrices above:
%                               b ->c1  c1 -> b
%                               b ->c2  c2 -> b
%                               b ->f1  f1 -> b
%                               b ->f2  f2 -> b


% Initialize random variables (Statistics Toolbox User Guide p. 2-11)

state = 137;
rand('state', state);
randn('state', state);


% Initialize strategy populations at equilibrium

nc0=fix(N*ones(1,2)/8); nc=nc0;      %round towards lower integer
nf0=ceil(N*ones(1,2)/8); nf=nf0;     %round towards higher integer
nb0=N-sum([nc0,nf0]); nb=nb0;        %rest is bondinvestors


% Initialization of traders aggregate cash for tc=1, Pf0=exp(pf0)=1

ccash = zeros(1,2); %(1*2) implies chartist start wealth = nc
fcash = zeros(1,2); %(1*2) fundamentalist's cash: no stocks at p=pf

cchist = zeros(T,2);    %(T*1) history of chartist aggregate cash
cfhist = zeros(T,2);    %(T*1) history of fundamentalist aggr. cash


%Simulation loop with price and population recording at integer steps
```

```
for t = 1:T %Start outer loop over integer time steps

    for st = 1:steps    %start inner loop over micro time steps

        % Calculate traders utilities

        c(1) = (nc(1)+nf(1)-nc(2)-nf(2))/N;
        c(2) = -c(1);       %(1*2) chartist utilities
        f = s*abs(pf-p);    %(1*2) fundamentalist utilities


        % Fill probability matrix for transitions within stocks

        pmat(1,1) = c(2) - c(1);    % c1 -> c2
        pmat(2,1) = f(2) - f(1);    % f1 -> f2
        pmat(3,1) = f(1) - c(1);    % c1 -> f1
        pmat(4,1) = f(2) - c(2);    % c2 -> f2
        pmat(5,1) = f(2) - c(1);    % c1 -> f2
        pmat(6,1) = f(1) - c(2);    % c2 -> f1
        pmat(:,2) = -pmat(:,1);     % reverse direction of col. 1

        pmat = (v/steps)*exp(a*pmat);   %(6*2) transition prob's


        % Fill prob-matrix for transitions between stocks / bonds

        ne = N - nb;                        %# equity investors
        pbe = exp(ab*(ne-nb)/N);            %p_BE: from bond to equity
        pbmat(1:2,1) = nc'*(pbe/ne);        % b -> c1; b -> c2
        pbmat(3:4,1) = nf'*(pbe/ne);        % b -> f1; b -> f2
        pbmat(:,2) = repmat(1/pbe,4,1); % c1, c2, f1, f2 -> b

        pbmat = (vb/steps)*pbmat;            %(4*2) transition prob's


        % (6*2) draws of traders leaving their strategy w'in stocks

        dnmat(1,1) = fastbin(nc(1), pmat(1,1)); % c1 -> c2
        dnmat(1,2) = fastbin(nc(2), pmat(1,2)); % c2 -> c1
        dnmat(2,1) = fastbin(nf(1), pmat(2,1)); % f1 -> f2
        dnmat(2,2) = fastbin(nf(2), pmat(2,2)); % f2 -> f1
        dnmat(3,1) = fastbin(nc(1), pmat(3,1)); % c1 -> f1
        dnmat(3,2) = fastbin(nf(1), pmat(3,2)); % f1 -> c1
        dnmat(4,1) = fastbin(nc(2), pmat(4,1)); % c2 -> f2
        dnmat(4,2) = fastbin(nf(2), pmat(4,2)); % f2 -> c2
        dnmat(5,1) = fastbin(nc(1), pmat(5,1)); % c1 -> f2
```

```
        dnmat(5,2) = fastbin(nf(2), pmat(5,2)); % f2 -> c1
        dnmat(6,1) = fastbin(nc(2), pmat(6,1)); % c2 -> f1
        dnmat(6,2) = fastbin(nf(1), pmat(6,2)); % f1 -> c2


        % (4*2) draws of traders switching between stocks / bonds

        dnbmat(1,1) = fastbin(nb, pbmat(1,1));   % b -> c1
        dnbmat(2,1) = fastbin(nb, pbmat(2,1));   % b -> c2
        dnbmat(3,1) = fastbin(nb, pbmat(3,1));   % b -> f1
        dnbmat(4,1) = fastbin(nb, pbmat(4,1));   % b -> f2

        dnbmat(1,2) = fastbin(nc(1), pbmat(1,2));   %c1 -> b
        dnbmat(2,2) = fastbin(nc(2), pbmat(2,2));   %c2 -> b
        dnbmat(3,2) = fastbin(nf(1), pbmat(3,2));   %f1 -> b
        dnbmat(4,2) = fastbin(nf(2), pbmat(4,2));   %f2 -> b


        % Calculation of stock population increments

        dnc(1)= dnmat(1,2) + dnmat(3,2) + dnmat(5,2) + dnbmat(1,1) - ...
                dnmat(1,1) - dnmat(3,1) - dnmat(5,1) - dnbmat(1,2);
        dnc(2)= dnmat(1,1) + dnmat(4,2) + dnmat(6,2) + dnbmat(2,1) - ...
                dnmat(1,2) - dnmat(4,1) - dnmat(6,1) - dnbmat(2,2);
        dnf(1)= dnmat(2,2) + dnmat(3,1) + dnmat(6,1) + dnbmat(3,1) - ...
                dnmat(2,1) - dnmat(3,2) - dnmat(6,2) - dnbmat(3,2);
        dnf(2)= dnmat(2,1) + dnmat(4,1) + dnmat(5,1) + dnbmat(4,1) - ...
                dnmat(2,2) - dnmat(4,2) - dnmat(5,2) - dnbmat(4,2);


        % Update of trader populations

        nc = nc + dnc;  %(1*2) updated chartist populations
        nf = nf + dnf;  %(1*2) updated fundamentalist populations
        nb = N - sum([nc nf]);  %(scalar) upd. bond population


        % Update of (1*2) trading prices

        dp = (l*dnc + (pf-p).*dnf)./nf; %(1*2) price increments
        p = p + dp;         %(1*2) updated trading prices


        % Update of traders cash

        price = exp(p);             %(1*2) ordinary trading price
        ccash = ccash-dnc.*price;   %(1*2) aggr. chartists cash
```

```
        fcash = fcash+dnc.*price;   %(1*2) aggr. fundament. cash


    end %end inner loop over micro time steps

    % Update history of state variables at integer time steps

    phist(t,:)=p;   %(T*2) history of logarithmic trading prices
    Phist(t,:)=price;   %(T*2) history of ordinary trading prices
    nchist(t,:)=nc; %(T*2) history of chartist populations
    nfhist(t,:)=nf; %(T*2) history of fundamentalist populations


    % Record history of cash

    cchist(t,:) = ccash;  %(scalar) chartists aggregate cash
    cfhist(t,:) = fcash;  %(scalar) fundamentalists aggr. cash


end %end outer loop over integer time steps


% Calculation of Return Series

%Individual logreturns
rhist = phist-[pf;phist(1:T-1,:)];  %(T*2) individual logreturns
corr = corrcoef(rhist); %(2*2) cross-correlation matrix of returns

%Equal weighted index returns
ret = exp(rhist);       %(T*2) individual gross returns
ewret = mean(ret,2)-1;  %(T*1) equally weighted index net-returns

%Capitalization weighted index returns
price = exp([pf; phist(1:T-1,:)]);  %(T*2) lagged price history
weight = price./repmat(sum(price,2),1,2);   %(T*2) cap. weigths
cwret = sum(weight.*ret,2)-1;   %(T*1) cap-weighted index net-returns

%Index logreturns
ewlret = log(1+ewret);  %(T*1) logreturns of equally weigthed index
cwlret = log(1+cwret);  %(T*1) logreturns of cap-weighted index
avlret = mean(rhist,2); %(T*1) average logreturns


% Calculate history of traders holdings normalized at tc=1

hchist = nchist;  %(T*2) chartist aggr. holdings history for tc=1
hfhist = nfhist.*(repmat(pf,T,1)-phist)/l;  %(T*2) fund. holdings
```

```
% Calculate history of traders wealth

cwealth = sum(hchist.*Phist+cchist,2);  %(T*1) chartists wealth
fwealth = sum(hfhist.*Phist+cfhist,2);  %(T*1) fundament. wealth


% Create graphical output

%Output 1: Trader Populations and Trading Prices
figure;
subplot(2,1,1),plot([nchist,nfhist]),title('Population Values');
subplot(2,1,2),plot(phist),title('Trading Prices');

%Output 2: Logreturns for asset 1
figure;
subplot(2,1,1),plot(rhist(:,1)),title({['Logreturns Asset 1'];...
    ['(Correlation with Asset 2 = ',num2str(corr(1,2)),')']});
subplot(2,1,2),
plot(100*[hchist(:,1)./sum([hchist(:,1),hfhist(:,1)],2),...
    hfhist(:,1)./sum([hchist(:,1),hfhist(:,1)],2)]),
title('Asset 1 holdings in % (Chartists blue, Fundamentalists green)');

%Output 3: Logreturns for asset 2
figure;
subplot(2,1,1),plot(rhist(:,2)),title({['Logreturns Asset 2'];...
    ['(Correlation with Asset 1 = ',num2str(corr(1,2)),')']});
subplot(2,1,2),
plot(100*[hchist(:,2)./sum([hchist(:,2),hfhist(:,2)],2),...
    hfhist(:,2)./sum([hchist(:,2),hfhist(:,2)],2)]),
title('Asset 2 holdings in % (Chartists blue, Fundamentalists green)');

%Output 4: Equal and capitalization weighted index logreturns
figure;
subplot(2,1,1),plot(ewlret),title('Equal Weighted Index Logreturns');
subplot(2,1,2),plot(cwlret),
title('Capitalization Weighted Index Logreturns');

%Output 5: Average Logreturn and holdings of the two stocks
figure;
subplot(2,1,1),
plot(avlret),title('Average Logreturn'),
subplot(2,1,2),
plot(100*[sum(hchist,2)./sum([hchist,hfhist],2),...
    sum(hfhist,2)./sum([hchist,hfhist],2)]),
title('Traders holdings in % (Chartists blue, Fundamentalists green)');
```

```
%Output 6: Traders cash and wealth
figure;
subplot(2,1,1),plot([sum(cchist,2),sum(cfhist,2)]),
title('Traders Cash (Chartists blue, Fundamentalists green)');
subplot(2,1,2), plot([cwealth,fwealth]);
title('Traders wealth (Chartists blue, Fundamentalists green)');


% Provide function output

stock1 = rhist(:,1);    %(T*1) logreturns of stock 1
stock2 = rhist(:,2);    %(T*1) logreturns of stock 2
ewindex = ewlret;       %(T*1) logreturns equal weighted index
cwindex = cwlret;       %(T*1) logreturns cap. weighted index

toc %end clock


function k = fastbin(n,p)
%FASTBIN generates binomial random variates optimized for speed
%K = FASTBIN(N,P) generates a random number from the
%binomial distribution with sample size N and probability P.
%Uses the BINV algorithm described in:
%Voratas Kachitvichyanukul and Bruce W. Schmeiser (1988):
%Binomial Random Variate Generation
%Communications of the ACM 31, 216-222
%
% written by Bernd Pape

    %BINV algorithm starts here
    %Step 1
    q = 1-p; s = p/q; a = (n+1)*s; r = q^n;

    %Step 2
    u = rand; k = 0;

    %Step 3 + 4
    while u > r
        u = u - r;
        k = k + 1;
        r = ((a/k)-s)*r;
    end
    %end of BINV algorithm
```

A7 Derivation of master and quasi-meanvalue equations for nearest neighbour transitions between arbitrarily many investment styles

The following derivation of the master and quasi-meanvalue equations for nearest neighbour transitions between arbitrarily many investment styles follwows rather closely the treatment of the corresponding concepts in chapters 10 and 11 of Weidlich (2002).

Consider a set of L possible investment styles $i = 1, 2, \ldots, L$, with n_i traders investing according to style i. The configuration of the investment community is then fully described by the vector $\mathbf{n} = \{n_1, \ldots, n_i, \ldots, n_L\}$ with $n_i \geq 0 \;\forall\; i \in \{1, \ldots, L\}$. Let $P(\mathbf{n}; t)$ denote the probability of finding the investment community in state $\mathbf{n}$ at time t. The probabilistic evolution of the configuration $\mathbf{n}$ between times t_1 and $t_2 = t_1 + \tau$ is given by the law of total probability as

$$P(\mathbf{n}; t_2) = \sum_{\{\mathbf{n}'\}} P(\mathbf{n}; t_2|\mathbf{n}', t_1)P(\mathbf{n}'; t_1). \tag{A7.1}$$

Therefore, the probability of observing $\mathbf{n}$ changes between t_1 and t_2 by

$$\begin{aligned} P(\mathbf{n}; t_2) - P(\mathbf{n}; t_1) &= \sum_{\mathbf{n}' \neq \mathbf{n}} P(\mathbf{n}; t_2|\mathbf{n}', t_1)P(\mathbf{n}'; t_1) + P(\mathbf{n}; t_2|\mathbf{n}, t_1)P(\mathbf{n}; t_1) - P(\mathbf{n}; t_1) \\ &= \sum_{\mathbf{n}' \neq \mathbf{n}} P(\mathbf{n}; t_2|\mathbf{n}', t_1)P(\mathbf{n}'; t_1) - (1 - P(\mathbf{n}; t_2|\mathbf{n}, t_1))P(\mathbf{n}; t_1) \\ &= \sum_{\mathbf{n}' \neq \mathbf{n}} P(\mathbf{n}; t_2|\mathbf{n}', t_1)P(\mathbf{n}'; t_1) - \sum_{\mathbf{n}' \neq \mathbf{n}} P(\mathbf{n}'; t_2|\mathbf{n}, t_1)P(\mathbf{n}; t_1) \end{aligned} \tag{A7.2}$$

due to the normalization condition $\sum_{\{\mathbf{n}'\}} P(\mathbf{n}'; t_2|\mathbf{n}, t_1) = 1$.
Expanding $P(\mathbf{n}; t_2|\mathbf{n}', t_1)$ and $P(\mathbf{n}'; t_2|\mathbf{n}, t_1)$ in a Taylors series around $t_1 = t$ with respect to $t_2 = t + \tau$ yields

$$\begin{aligned} P(\mathbf{n}; t_2|\mathbf{n}', t_1) &= P(\mathbf{n}; t|\mathbf{n}; t) + \tau \left. \frac{\partial P(\mathbf{n}; t_2|\mathbf{n}', t_1)}{\partial t_2} \right|_{t_2 = t} + o(\tau^2) \\ &= \delta_{\mathbf{n},\mathbf{n}'} + \tau w(\mathbf{n}|\mathbf{n}') + o(\tau^2) \end{aligned} \tag{A7.3a}$$

and similarly

$$P(\mathbf{n}'; t_2|\mathbf{n}, t_1) = \delta_{\mathbf{n}',\mathbf{n}} + \tau w(\mathbf{n}'|\mathbf{n}) + o(\tau^2), \tag{A7.3b}$$

where δ denotes the Kronecker delta function defined in equation (4.22), $w(\mathbf{n}'|\mathbf{n})$ denotes the per unit time transition probability between configurations $\mathbf{n}$ and $\mathbf{n}'$ as in

(4.20), and $o(\tau^2)$ stands for terms of second and higher order in τ. Inserting (A7.3) into (A7.2) and taking the limit $\tau \to 0$ yields the fundamental master equation for the probability change over infinitesimally small time intervals

$$\dot{P}(\mathbf{n};t) := \frac{d\,P(\mathbf{n};t)}{dt} = \lim_{t_2 \to t_1} \frac{P(\mathbf{n};t_2) - P(\mathbf{n};t_1)}{t_2 - t_1}$$

as

$$\dot{P}(\mathbf{n};t) = \sum_{\mathbf{n}' \neq \mathbf{n}} w(\mathbf{n}|\mathbf{n}')P(\mathbf{n}';t) - \sum_{\mathbf{n}' \neq \mathbf{n}} w(\mathbf{n}'|\mathbf{n})P(\mathbf{n};t). \tag{A7.4}$$

In the next step we confine ourselves to transitions between neighbouring states of the investment configuration as a result of the Poisson-type dynamics induced by at most one trader changing her strategy during any infinitesimal time interval τ.[144] That is, we consider only transitions between configurations $\mathbf{n}$ and

$$\mathbf{n}_{ij} := \{n_1, \ldots, (n_i - 1), \ldots, (n_j + 1), \ldots, n_L\},$$

such that

$$w(\mathbf{n}'|\mathbf{n}) = w(\mathbf{n}|\mathbf{n}') = 0 \text{ for } \ \mathbf{n}' \neq \mathbf{n}_{ij},$$

and define

$$w_{ij}(\mathbf{n}) := w(\mathbf{n}_{ij}|\mathbf{n}) = n_i p_{ij} \tag{A7.5}$$

in line with (4.21), (4.23) and (4.24), with p_{ij} denoting the probability for a single trader to change from strategy i to strategy j. The fundamental master equation (A7.4) reduces then to

$$\begin{aligned}\dot{P}(\mathbf{n};t) &= \sum_{i \neq j}^{L} w_{ji}(\mathbf{n}_{ij})P(\mathbf{n}_{ij};t) - \sum_{i \neq j}^{L} w_{ij}(\mathbf{n})P(\mathbf{n};t) \\ &= \sum_{i,j=1}^{L} w_{ji}(\mathbf{n}_{ij})P(\mathbf{n}_{ij};t) - \sum_{i,j=1}^{L} w_{ij}(\mathbf{n})P(\mathbf{n};t) \end{aligned} \tag{A7.6}$$

Note that we need not exclude $i = j$ because $\mathbf{n}_{ii} = \mathbf{n}$.

Introduce next translation operators T_i^+ and T_i^- on the configuration space $\{\mathbf{n}\}$ as

$$T_i^{\pm} F(n_1, \ldots, n_i, \ldots, n_L) := F(n_1, \ldots, (n_i \pm 1), \ldots, n_L), \tag{A7.7}$$

[144] see the discussion of the herding model by Lux (1995) on pp. 78.

such that

$$T_j^+ T_i^- F(\mathbf{n}) = F(\mathbf{n}_{ij}) \quad \text{and} \quad T_i^+ T_j^- F(\mathbf{n}) = F(\mathbf{n}_{ji}). \tag{A7.8}$$

We may therefore write

$$\sum_{i,j=1}^{L} T_i^+ T_j^- w_{ij}(\mathbf{n}) P(\mathbf{n};t) = \sum_{i,j=1}^{L} w_{ij}(\mathbf{n}_{ji}) P(\mathbf{n}_{ji};t) = \sum_{i,j=1}^{L} w_{ji}(\mathbf{n}_{ij}) P(\mathbf{n}_{ij};t),$$

such that the master equation (A7.6) may be rewritten as

$$\dot{P}(\mathbf{n};t) = \sum_{i,j=1}^{L} (T_i^+ T_j^- - 1) w_{ij}(\mathbf{n}) P(\mathbf{n};t). \tag{A7.9}$$

Note the following properties of the translation operators:

$$\begin{aligned}
& & T_i^{\pm} n_k F(\mathbf{n}) &= (n_k \pm \delta_{ik}) T_i^{\pm} F(\mathbf{n}) & \text{(A7.10a)} \\
&\Rightarrow & n_k T_i^{\pm} F(\mathbf{n}) &= T_i^{\pm} n_k F(\mathbf{n}) \mp \delta_{ik} T_i^{\pm} F(\mathbf{n}) = T_i^{\pm} (n_k \mp \delta_{ik}) F(\mathbf{n}) & \text{(A7.10b)} \\
&\Rightarrow & n_k T_i^+ T_j^- F(\mathbf{n}) &= T_i^+ n_k T_j^- F(\mathbf{n}) - T_i^+ \delta_{ik} T_j^- F(\mathbf{n}) & \\
& & &= T_i^+ T_j^- (n_k + \delta_{jk}) F(\mathbf{n}) - T_i^+ T_j^- \delta_{ik} F(\mathbf{n}) & \\
& & &= T_i^+ T_j^- (n_k + \delta_{jk} - \delta_{ik}) F(\mathbf{n}) & \text{(A7.10c)}
\end{aligned}$$

Property (A7.10c) may be used in conjunction with (A7.9) in order to calculate the dynamics of the expected number of traders using strategy k,

$$<n_k>_t := \sum_{\{\mathbf{n}\}} n_k P(\mathbf{n};t),$$

as

$$
\begin{aligned}
<\dot{n}_k>_t &= \sum_{\{\mathbf{n}\}} n_k \dot{P}(\mathbf{n};t) \\
&= \sum_{i,j=1}^{L} \sum_{\{\mathbf{n}\}} n_k (T_i^+ T_j^- - 1) w_{ij}(\mathbf{n}) P(\mathbf{n};t) \\
&= \sum_{i,j=1}^{L} \sum_{\{\mathbf{n}\}} T_i^+ T_j^- (\delta_{jk} - \delta_{ik}) w_{ij}(\mathbf{n}) P(\mathbf{n};t) \\
&= \sum_{i=1}^{L} \sum_{\{\mathbf{n}\}} w_{ik}(\mathbf{n}_{ki}) P(\mathbf{n}_{ki};t) - \sum_{j=1}^{L} \sum_{\{\mathbf{n}\}} w_{kj}(\mathbf{n}_{jk}) P(\mathbf{n}_{jk};t) \\
&= \sum_{i=1}^{L} \sum_{\{\mathbf{n}\}} w_{ik}(\mathbf{n}) P(\mathbf{n};t) - \sum_{j=1}^{L} \sum_{\{\mathbf{n}\}} w_{kj}(\mathbf{n}) P(\mathbf{n};t) \\
&= \sum_{i=1}^{L} <w_{ik}(\mathbf{n})>_t - \sum_{j=1}^{L} <w_{kj}(\mathbf{n})>_t \\
&= \sum_{i=1}^{L} <w_{ik}(\mathbf{n}) - w_{ki}(\mathbf{n})>_t. \qquad \text{(A7.11)}
\end{aligned}
$$

Equation (A7.11) is the exact mean value equation for trader population n_k. In order to obtain the approximate quasi-meanvalue equations for the trader populations, expand all transition rates $w_{ij}(\mathbf{n})$ to first order around their values at the expected configuration $<\mathbf{n}>_t$ at time t,

$$
w_{ij}(\mathbf{n}) \approx w_{ij}(<\mathbf{n}>_t) + \sum_{l=1}^{L} \frac{\partial\, w_{ij}(<\mathbf{n}>_t)}{\partial n_l} \Delta n_l, \qquad \text{(A7.12)}
$$

and insert into the exact mean value equations (A7.11):

$$
\begin{aligned}
<\dot{n}_k>_t &\approx \sum_{i=1}^{L} [<w_{ik}(<\mathbf{n}>_t) - w_{ki}(<\mathbf{n}>_t)>_t] \\
&\quad + \sum_{i=1}^{L} \sum_{l=1}^{L} \left(\frac{\partial\, w_{ik}(<\mathbf{n}>_t)}{\partial n_l} - \frac{\partial\, w_{ki}(<\mathbf{n}>_t)}{\partial n_l} \right) <\Delta n_l>_t \\
&= \sum_{i=1}^{L} [w_{ik}(<\mathbf{n}>_t) - w_{ki}(<\mathbf{n}>_t)] \qquad \text{(A7.13)}
\end{aligned}
$$

Inserting the individual transition probablilites (A7.5) yields the quasi-meanvalue equations in the form of the main text:

$$
<\dot{n}_k>_t = \sum_{i=1}^{L} \left(<n_i>_t\, p_{ik} - <n_k>_t\, p_{ki} \right). \qquad \text{(A7.14)}
$$

A8 Proof of Proposition 1

A fundamental equilibrium requires

$$\dot{n_{c1}} = \dot{n_{c2}} = \dot{n_{f1}} = \dot{n_{f2}} = 0 \quad \text{at} \quad p_1 \equiv p_{f1} \text{ and } p_2 \equiv p_{f2}. \tag{A8.1}$$

Using the identity

$$\begin{aligned} & n_j e^{\alpha(U_i - U_j)} - n_i e^{\alpha(U_j - U_i)} \\ =& (n_i + n_j) \cdot \left[\tanh(\alpha(U_i - U_j)) - \frac{n_i - n_j}{n_i + n_j} \right] \cosh(\alpha(U_i - U_j)) \end{aligned} \tag{A8.2}$$

the equations of motion for the trader populations (5.14) may be rewritten in terms of hyperbolic funcions as

$$\begin{aligned} \dot{n_{c1}} = & v_B \; n_{c1} \left[\frac{n_B}{n_E} e^{\alpha_B (n_E - n_B)/N} - e^{-\alpha_B (n_E - n_B)/N} \right] \\ & + v \Bigg\{ \; (n_{c1} + n_{c2}) \left[\tanh(\alpha(C_1 - C_2)) - \frac{n_{c1} - n_{c2}}{n_{c1} + n_{c2}} \right] \cosh(\alpha(C_1 - C_2)) \\ & \quad + (n_{c1} + n_{f1}) \left[\tanh(\alpha(C_1 - F_1)) - \frac{n_{c1} - n_{f1}}{n_{c1} + n_{f1}} \right] \cosh(\alpha(C_1 - F_1)) \\ & \quad + (n_{c1} + n_{f2}) \left[\tanh(\alpha(C_1 - F_2)) - \frac{n_{c1} - n_{f2}}{n_{c1} + n_{f2}} \right] \cosh(\alpha(C_1 - F_2)) \Bigg\} \end{aligned} \tag{A8.3a}$$

$$\begin{aligned} \dot{n_{c2}} = & v_B \; n_{c2} \left[\frac{n_B}{n_E} e^{\alpha_B (n_E - n_B)/N} - e^{-\alpha_B (n_E - n_B)/N} \right] \\ & + v \Bigg\{ \; (n_{c2} + n_{c1}) \left[\tanh(\alpha(C_2 - C_1)) - \frac{n_{c2} - n_{c1}}{n_{c2} + n_{c1}} \right] \cosh(\alpha(C_2 - C_1)) \\ & \quad + (n_{c2} + n_{f1}) \left[\tanh(\alpha(C_2 - F_1)) - \frac{n_{c2} - n_{f1}}{n_{c2} + n_{f1}} \right] \cosh(\alpha(C_2 - F_1)) \\ & \quad + (n_{c2} + n_{f2}) \left[\tanh(\alpha(C_2 - F_2)) - \frac{n_{c2} - n_{f2}}{n_{c2} + n_{f2}} \right] \cosh(\alpha(C_2 - F_2)) \Bigg\} \end{aligned} \tag{A8.3b}$$

$$\begin{aligned} \dot{n_{f1}} = & v_B \; n_{f1} \left[\frac{n_B}{n_E} e^{\alpha_B (n_E - n_B)/N} - e^{-\alpha_B (n_E - n_B)/N} \right] \\ & + v \Bigg\{ \; (n_{f1} + n_{c1}) \left[\tanh(\alpha(F_1 - C_1)) - \frac{n_{f1} - n_{c1}}{n_{f1} + n_{c1}} \right] \cosh(\alpha(F_1 - C_1)) \\ & \quad + (n_{f1} + n_{c2}) \left[\tanh(\alpha(F_1 - C_2)) - \frac{n_{f1} - n_{c2}}{n_{f1} + n_{c2}} \right] \cosh(\alpha(F_1 - C_2)) \\ & \quad + (n_{f1} + n_{f2}) \left[\tanh(\alpha(F_1 - F_2)) - \frac{n_{f1} - n_{f2}}{n_{f1} + n_{f2}} \right] \cosh(\alpha(F_1 - F_2)) \Bigg\} \end{aligned} \tag{A8.3c}$$

$$
\begin{aligned}
\dot{n_{f2}} = v_B\, n_{f2} &\left[\frac{n_B}{n_E} e^{\alpha_B(n_E-n_B)/N} - e^{-\alpha_B(n_E-n_B)/N}\right] \\
&+ v \Bigg\{ (n_{f2}+n_{c1}) \left[\tanh(\alpha(F_2-C_1)) - \frac{n_{f2}-n_{c1}}{n_{f2}+n_{c1}}\right] \cosh(\alpha(F_2-C_1)) \\
&\quad + (n_{f2}+n_{c2}) \left[\tanh(\alpha(F_2-C_2)) - \frac{n_{f2}-n_{c2}}{n_{f2}+n_{c2}}\right] \cosh(\alpha(F_2-C_2)) \\
&\quad + (n_{f2}+n_{f1}) \left[\tanh(\alpha(F_2-F_1)) - \frac{n_{f2}-n_{f1}}{n_{f2}+n_{f1}}\right] \cosh(\alpha(F_2-F_1)) \Bigg\}
\end{aligned}
\tag{A8.3d}
$$

In order to fulfil the condition (A8.1) it suffices that all squared brackets above equal zero. That is the case when both

$$n_{c1} = n_{c2} = n_{f1} = n_{f2} = n_E/4 \quad \text{and} \quad n_B = n_E = N/2, \quad \text{as claimed.}$$

A9 Proof of Proposition 2

Local stability with respect to regime-specific dynamics will be considered by inspecting the Jacobian of the system of differential equations for the trader populations and prices.[145] We will for that purpose reformulate the population dynamics (A8.3) in terms of the new variables

$$c_1 := \frac{n_{c1}}{N}, \quad c_2 := \frac{n_{c2}}{N}, \quad f_1 := \frac{n_{f1}}{N}, \quad f_2 := \frac{n_{f2}}{N} \tag{A9.1}$$

as

$$\begin{aligned}\dot{c_1} = v_B\ c_1 & \left[\tfrac{1-c_1-c_2-f_1-f_2}{c_1+c_2+f_1+f_2} e^{\alpha_B(2(c_1+c_2+f_1+f_2)-1)} - e^{-\alpha_B(2(c_1+c_2+f_1+f_2)-1)}\right] \\ + v\Bigg\{ & (c_1+c_2)\left[\tanh(\alpha(C_1-C_2)) - \frac{c_1-c_2}{c_1+c_2}\right]\cosh(\alpha(C_1-C_2)) \\ & + (c_1+f_1)\left[\tanh(\alpha(C_1-F_1)) - \frac{c_1-f_1}{c_1+f_1}\right]\cosh(\alpha(C_1-F_1)) \\ & + (c_1+f_2)\left[\tanh(\alpha(C_1-F_2)) - \frac{c_1-f_2}{c_1+f_2}\right]\cosh(\alpha(C_1-F_2))\Bigg\}\end{aligned} \tag{A9.2a}$$

$$\begin{aligned}\dot{c_2} = v_B\ c_2 & \left[\tfrac{1-c_1-c_2-f_1-f_2}{c_1+c_2+f_1+f_2} e^{\alpha_B(2(c_1+c_2+f_1+f_2)-1)} - e^{-\alpha_B(2(c_1+c_2+f_1+f_2)-1)}\right] \\ + v\Bigg\{ & (c_2+c_1)\left[\tanh(\alpha(C_2-C_1)) - \frac{c_2-c_1}{c_2+c_1}\right]\cosh(\alpha(C_2-C_1)) \\ & + (c_2+f_1)\left[\tanh(\alpha(C_2-F_1)) - \frac{c_2-f_1}{c_2+f_1}\right]\cosh(\alpha(C_2-F_1)) \\ & + (c_2+f_2)\left[\tanh(\alpha(C_2-F_2)) - \frac{c_2-f_2}{c_2+f_2}\right]\cosh(\alpha(C_2-F_2))\Bigg\}\end{aligned} \tag{A9.2b}$$

[145] see e.g. Gandolfo (1996).

$$\begin{aligned}\dot{f}_1 = v_B\ f_1 &\left[\tfrac{1-c_1-c_2-f_1-f_2}{c_1+c_2+f_1+f_2} e^{\alpha_B(2(c_1+c_2+f_1+f_2)-1)} - e^{-\alpha_B(2(c_1+c_2+f_1+f_2)-1)}\right] \\ + v \Bigg\{ &(f_1+c_1)\left[\tanh(\alpha(F_1-C_1)) - \frac{f_1-c_1}{f_1+c_1}\right]\cosh(\alpha(F_1-C_1)) \\ &+ (f_1+c_2)\left[\tanh(\alpha(F_1-C_2)) - \frac{f_1-c_2}{f_1+c_2}\right]\cosh(\alpha(F_1-C_2)) \\ &+ (f_1+f_2)\left[\tanh(\alpha(F_1-F_2)) - \frac{f_1-f_2}{f_1+f_2}\right]\cosh(\alpha(F_1-F_2))\Bigg\}\end{aligned} \tag{A9.2c}$$

$$\begin{aligned}\dot{f}_2 = v_B\ f_2 &\left[\tfrac{1-c_1-c_2-f_1-f_2}{c_1+c_2+f_1+f_2} e^{\alpha_B(2(c_1+c_2+f_1+f_2)-1)} - e^{-\alpha_B(2(c_1+c_2+f_1+f_2)-1)}\right] \\ + v \Bigg\{ &(f_2+c_1)\left[\tanh(\alpha(F_2-C_1)) - \frac{f_2-c_1}{f_2+c_1}\right]\cosh(\alpha(F_2-C_1)) \\ &+ (f_2+c_2)\left[\tanh(\alpha(F_2-C_2)) - \frac{f_2-c_2}{f_2+c_2}\right]\cosh(\alpha(F_2-C_2)) \\ &+ (f_2+f_1)\left[\tanh(\alpha(F_2-F_1)) - \frac{f_2-f_1}{f_2+f_1}\right]\cosh(\alpha(F_2-F_1))\Bigg\} \\ .\end{aligned} \tag{A9.2d}$$

The price dynamics (5.3) may be rewritten in terms of (A9.1) as

$$\dot{p}_i = \frac{1}{f_i}\left(l\dot{c}_i + (p_{fi}-p_i)\dot{f}_i\right), \qquad i = 1, 2, \tag{A9.3}$$

where we have used the leverage parameter $l := t_c/t_f$ introduced in proposition 2 in order to express the relation of chartist relative to fundamentalist target holdings. The entries of the Jakobian matrix

$$J = \begin{pmatrix} \frac{\partial \dot{c}_1}{\partial c_1} & \frac{\partial \dot{c}_1}{\partial c_2} & \frac{\partial \dot{c}_1}{\partial f_1} & \frac{\partial \dot{c}_1}{\partial f_2} & \frac{\partial \dot{c}_1}{\partial p_1} & \frac{\partial \dot{c}_1}{\partial p_2} \\ \frac{\partial \dot{c}_2}{\partial c_1} & \frac{\partial \dot{c}_2}{\partial c_2} & \frac{\partial \dot{c}_2}{\partial f_1} & \frac{\partial \dot{c}_2}{\partial f_2} & \frac{\partial \dot{c}_2}{\partial p_1} & \frac{\partial \dot{c}_2}{\partial p_2} \\ \frac{\partial \dot{f}_1}{\partial c_1} & \frac{\partial \dot{f}_1}{\partial c_2} & \frac{\partial \dot{f}_1}{\partial f_1} & \frac{\partial \dot{f}_1}{\partial f_2} & \frac{\partial \dot{f}_1}{\partial p_1} & \frac{\partial \dot{f}_1}{\partial p_2} \\ \frac{\partial \dot{f}_2}{\partial c_1} & \frac{\partial \dot{f}_2}{\partial c_2} & \frac{\partial \dot{f}_2}{\partial f_1} & \frac{\partial \dot{f}_2}{\partial f_2} & \frac{\partial \dot{f}_2}{\partial p_1} & \frac{\partial \dot{f}_2}{\partial p_2} \\ \frac{\partial \dot{p}_1}{\partial c_1} & \frac{\partial \dot{p}_1}{\partial c_2} & \frac{\partial \dot{p}_1}{\partial f_1} & \frac{\partial \dot{p}_1}{\partial f_2} & \frac{\partial \dot{p}_1}{\partial p_1} & \frac{\partial \dot{p}_1}{\partial p_2} \\ \frac{\partial \dot{p}_2}{\partial c_1} & \frac{\partial \dot{p}_2}{\partial c_2} & \frac{\partial \dot{p}_2}{\partial f_1} & \frac{\partial \dot{p}_2}{\partial f_2} & \frac{\partial \dot{p}_2}{\partial p_1} & \frac{\partial \dot{p}_2}{\partial p_2} \end{pmatrix} \tag{A9.4}$$

of the coupled system (A9.2) and (A9.3) evaluated at the fundamental equilibrium

$$c_1 = c_2 = f_1 = f_2 = 1/8 \tag{A9.5}$$

read for the population subdynamics

$$\frac{\partial \dot{c}_1}{\partial c_1} = \frac{\partial \dot{c}_2}{\partial c_2} = (\alpha - 3)v + (\alpha_B - 1)\frac{v_B}{2} \tag{A9.6a}$$

$$\frac{\partial \dot{c}_1}{\partial c_2} = \frac{\partial \dot{c}_1}{\partial f_2} = \frac{\partial \dot{c}_2}{\partial c_1} = \frac{\partial \dot{c}_2}{\partial f_1} = (\alpha - 1)v + (\alpha_B - 1)\frac{v_B}{2} \tag{A9.6b}$$

$$\frac{\partial \dot{c}_1}{\partial f_1} = \frac{\partial \dot{c}_2}{\partial f_2} = (\alpha + 1)v + (\alpha_B - 1)\frac{v_B}{2} \tag{A9.6c}$$

$$\frac{\partial \dot{c}_1}{\partial p_1} = \frac{\partial \dot{c}_2}{\partial p_1} = \frac{\partial \dot{f}_2}{\partial p_1} = -\frac{\alpha v}{4} F_1{}'(p_{f1}) \tag{A9.6d}$$

$$\frac{\partial \dot{c}_1}{\partial p_2} = \frac{\partial \dot{c}_2}{\partial p_2} = \frac{\partial \dot{f}_1}{\partial p_2} = -\frac{\alpha v}{4} F_2{}'(p_{f2}) \tag{A9.6e}$$

$$\frac{\partial \dot{f}_1}{\partial c_1} = \frac{\partial \dot{f}_1}{\partial c_2} = \frac{\partial \dot{f}_1}{\partial f_2} = \frac{\partial \dot{f}_2}{\partial c_1} = \frac{\partial \dot{f}_2}{\partial c_2} = \frac{\partial \dot{f}_2}{\partial f_1} = v + (\alpha_B - 1)\frac{v_B}{2} \tag{A9.6f}$$

$$\frac{\partial \dot{f}_1}{\partial f_1} = \frac{\partial \dot{f}_2}{\partial f_2} = (\alpha_B - 1)\frac{v_B}{2} - 3v \tag{A9.6g}$$

$$\frac{\partial \dot{f}_1}{\partial p_1} = \frac{3}{4}\alpha v F_1{}'(p_{f1}) \tag{A9.6h}$$

$$\frac{\partial \dot{f}_2}{\partial p_2} = \frac{3}{4}\alpha v F_2{}'(p_{f2}) \tag{A9.6i}$$

Application of the chain rule to (A9.3) at $p_{1/2} = p_{f1/2}$ yields for the price subdynamics

$$\frac{\partial \dot{p}_1}{\partial c_1} = 8l\frac{\partial \dot{c}_1}{\partial c_1}, \quad \frac{\partial \dot{p}_1}{\partial c_2} = 8l\frac{\partial \dot{c}_1}{\partial c_2}, \quad \dots \quad \frac{\partial \dot{p}_2}{\partial p_2} = 8l\frac{\partial \dot{c}_2}{\partial p_2}. \tag{A9.7}$$

A complication arises from the fact that $F_1{}'(p_{f1})$ and $F_2{}'(p_{f2})$ are not defined because

$$F_{1/2}(p_{1/2}) = s|p_{f1/2} - p_{1/2}| \tag{A9.8}$$

implies a jump of the derivative $F'_{1/2}$ at the respective fundamental price

$$F_{1/2}{}'(p_{1/2}) = \pm s \qquad \text{for} \qquad p_{f1/2} \lesseqgtr p_{1/2}. \tag{A9.9}$$

It is therefore necessary to examine each of the regimes ($p_1 > p_{f1}$, $p_2 > p_{f2}$), ($p_1 < p_{f1}$, $p_2 < p_{f2}$), and ($p_1 \gtrless p_{f1}$, $p_2 \lessgtr p_{f2}$) separately. Furthermore, stability with respect to regime-specific dynamics is in general neither a sufficient nor necessary condition for stability of the overall dynamics (Honkapohja & Ito 1983). The following analysis serves therefore only as a general guideline, which factors may have an impact upon local stability of the fundamental equilibrium within the overall dynamics. All four regimes share the common eigenvalues:

$$\lambda_1 = \lambda_2 = 0, \quad \text{and} \tag{A9.10a}$$

$$\lambda_3 = 2(\alpha_B - 1)v_b. \tag{A9.10b}$$

The remaining eigenvalues differ from one regime to another. Consider first the case where the mispricings in both stocks have the same sign:

$$\lambda_{4,\pm\pm} = -4v(1 \pm \alpha ls), \tag{A9.11a}$$

$$\lambda_{5,\pm\pm} = \quad v\left[\left(1 + \sqrt{1 \pm 16ls}\right)\alpha - 4\right], \tag{A9.11b}$$

$$\lambda_{6,\pm\pm} = \quad v\left[\left(1 - \sqrt{1 \pm 16ls}\right)\alpha - 4\right], \tag{A9.11c}$$

where the plus sigs apply to $(p_1 > p_{f1},\, p_2 > p_{f2})$ and the minus signs to the regime $(p_1 < p_{f1},\, p_2 < p_{f2})$. The last three eigenvalues for the regimes $(p_1 \gtrless p_{f1},\, p_2 \lessgtr p_{f2})$ read

$$\lambda_{4,\pm\mp} = \frac{2}{3}v\left\{\alpha\left[1 + f(ls)^{1/3} + \frac{1}{f(ls)^{1/3}}\right] - 6\right\}, \tag{A9.12a}$$

$$\lambda_{5/6,\pm\mp} = \frac{1}{3}v\left\{\alpha\left[\left(2 - f(ls)^{1/3} - \frac{1}{f(ls)^{1/3}}\right) \pm i\sqrt{3}\left(f(ls)^{1/3} - \frac{1}{f(ls)^{1/3}}\right)\right] - 12\right\}, \tag{A9.12b}$$

$$\text{with} \qquad f(ls) := 1 - 108(ls)^2 + 6ls\sqrt{324(ls)^2 - 6}, \tag{A9.12c}$$

the real parts of which are given by

$$\mathrm{Re}(\lambda_{4,\pm\mp}) = \frac{2}{3}v\left\{\alpha\left[1 + \left(|f(ls)|^{1/3} + \frac{1}{|f(ls)|^{1/3}}\right)\cos\left(\frac{1}{3}\arg(f(ls))\right)\right] - 6\right\}, \tag{A9.13a}$$

$$\begin{aligned}\mathrm{Re}(\lambda_{5/6,\pm\mp}) = \frac{1}{3}v\Bigg\{\alpha\Bigg[2 - &\left(|f(ls)|^{1/3} + \frac{1}{|f(ls)|^{1/3}}\right)\cdot \\ &\left(\cos\left(\frac{1}{3}\arg(f(ls))\right) \pm \sqrt{3}\sin\left(\frac{1}{3}\arg(f(ls))\right)\right)\Bigg] - 12\Bigg\},\end{aligned} \tag{A9.13b}$$

none of which exceed

$$\mathrm{Re}(\lambda_{\pm\mp})_{\max} = \frac{2}{3}v\left\{\alpha\left[1 + |f(ls)|^{1/3} + \frac{1}{|f(ls)|^{1/3}}\right] - 6\right\}, \tag{A9.14}$$

which is again smaller than the largest eigenvalues in those regimes where the signs of the mispricings in both stocks coincide,

$$\mathrm{Re}(\lambda_{\pm\pm})_{\max} = \begin{cases} v\left[\left(1+\sqrt{1+16ls}\right)\alpha - 4\right], & \text{for } ls \leq 3/2, \\ 4v(\alpha ls - 1), & \text{for } ls \geq 3/2. \end{cases} \tag{A9.15}$$

The necessary conditions for stability with respect to the regime-specific dynamics listed in proposition 2 follow then from requiring the real part of all eigenvalues not to exceed zero in any of the four regimes, that is $\lambda_3 \leq 0$ and $\mathrm{Re}(\lambda_{\pm\pm})_{\max} \leq 0$.