# Algebra II (2008)

Exercise 5/week 46

1. Let $F/M$ and $M/K$ be finite field extensions. Show that $F/K$ is finite, and
$$[F : K] = [F : M][M : K].$$
2. Find the minimal polynomial of $\sqrt{2}+i$ over $\mathbb{Q}$ where $i$ is the imaginary unit. Show that $\mathbb{Q}(\sqrt{2}, i)) = \mathbb{Q}(\sqrt{2}+i)$.
3. Find the splitting field of $x^4 - 2x^3 + x - 2$ over $\mathbb{Q}$.
4. (a) Let $f$ be a polynomial over a field $F$. Show that $f$ has a multiple root in its splitting field over $F$ if and only if $\gcd(f, f') \neq 1$
   (b) Let $z \in \mathbb{C}$ algebraic over $\mathbb{Q}$. Show that the minimal polynomial of $z$ over $\mathbb{Q}$ has no multiple roots.
5. Let $\alpha$ be a root of the irreducible polynomial $x^4 + x^3 + x^2 + x + 1 \in \mathbb{F}_2[x]$ and let $\gamma$ be a root of the irreducible polynomial $x^4 + x + 1 \in \mathbb{F}_2[x]$.
   (a) Is $\gamma$ a primitive element of $\mathbb{F}_{16}$?
   (b) Find an isomorphism from $\mathbb{F}_2(\alpha)$ onto $\mathbb{F}_2(\gamma)$.
6. Find all the subfields of $\mathbb{F}_2(\alpha)$ and give for each of them a basis over $\mathbb{F}_2$.
7. Let $a \in \mathbb{F}_q$ and let $k$ be a positive integer. Find the number of solutions of $x^k = a$ in $\mathbb{F}_q$.
8. Prove: $x^2 = -1$ is solvable in $\mathbb{F}_q$ if and only if $q$ is even or $q \equiv 1\ (4)$.