Algebra II (2008)

Exercise 3/week 44

1. (a) How many subgroups does $\mathbb{Z}_{101}^*$ have?
   (b) Find all the subgroups of $\mathbb{Z}_{101}^*$ having 25 elements.
   (c) Find all the subgroups of $\mathbb{Z}_{101}^*$ having 8 elements.
2. Find all group homomorphisms from $\mathbb{Z}_9$ to $\mathbb{Z}_{10}$.
3. Let $(G, \cdot)$ ja $(G', \cdot)$ be groups. Show that $(G \times G', \cdot)$ is a group, the *direct product* of G and G', when $(a, b) \cdot (c, d) = (ac, bd)$ for all $a, c \in G$ and $b, d \in G'$.
4. Let m and n be positive integers such that $\gcd(m, n) = 1$. Prove:
$$\mathbb{Z}_{mn}^* \simeq \mathbb{Z}_m^* \times \mathbb{Z}_n^*.$$
5. Define the *Euler's function* $\phi : \mathbb{N} \to \mathbb{N}$, $\phi(n) = |\mathbb{Z}_n^*|$. Let $n = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ be the canonical prime decomposition of $n \in \mathbb{N}$ (i.e. the primes p_i satisfy $p_1 < p_2 < \cdots < p_t$, and $e_i > 0$ for all $i = 1, \ldots t$). Prove:
$$\phi(n) = (p_1 - 1)p_1^{e_1-1} \cdots (p_t - 1)p_t^{e_t-1}.$$
6. Let R be a ring, and let $a, b \in R$. Then
   (a) $0 \cdot a = 0$.
   (b) $1 \neq 0$.
   (c) $(-a)b = -ab$.
   (d) $(-a)(-b) = ab$.
7. Find all the ideals of $\mathbb{Z}$.
8. (a) Let R be a commutative ring, and let $a, b \in R$. Let $n \in \mathbb{N}$. Prove the Newton's binomial formula:
$$(a + b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$
   (b) Assume that the characteristic p of R is positive. Prove
$$(a + b)^{p^n} = a^{p^n} + b^{p^n}.$$