ON THE CONCEPTS AND DERIVATION OF RELIABILITY IN STOCHASTIC SYSTEMS WITH STATES OF REDUCED EFFICIENCY; AN APPLICATION OF SUPPLEMENTARY VARIABLES AND DISCRETE TRANSFORMS

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ACADEMIC DISSERTATION

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To Antti and Olli,
my own stochastic systems

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Ilkka Virtanen

CONTENTS

1 INTRODUCTION

11 Reliability as a part of the efficiency of a production system

Numerous problems concerning the improvement of the efficiency of a production system have acquired special importance in the last few decades. Central among these efficiency problems is that of the ability of the system to function in the way intended by the user. This general property of the system is commonly called the 'reliability of the system'.

The importance of reliability has grown in proportion as equipment has been automated, become more complex in construction and begun to assume responsibility for ever larger overall tasks without the continuous and direct control of man. Failures resulting in the breakdown of a system or other imperfections in its operation, i.e. the unreliability of the system, now have a number of injurious consequences for the system itself, the user, and the environment. These consequences are most clearly revealed in increased cost, wasted time, psychological effects of inconvenience, and in certain instances personal and national security.[1]

The cost of unreliability is not only the cost of the failing item (the cost of a new item and/or the cost of repairs) but also of additional consequences. An interruption in the operation of a device can also lead to deterioration in the quality of the manufactured product or to damage to or destruction of the related equipment. Further, costs are incurred by the

1 LLOYD and LIPOW (1962), pp. 1-3.

necessity of having trained personnel for checking equipment and by the necessity of having more items (spares) available than would be theoretically necessary if items were completely reliable.

Wasted time is a frequent consequence of unreliability. The time actually wasted is almost always longer than that required for repair or replacement, due to waiting, planning and checking. Time wasted, especially in industry, is very often synonymous with money wasted: there are production losses, late deliveries etc. resulting from the failure in question.

The psychological effect of unreliability is evident over a very wide area, ranging from the military sector to the commercial field. In the commercial field, for example, frequent or unpleasant inconveniences resulting from the unreliability of a device will soon bring customer dissatisfaction and loss of business.

Lastly, perhaps the most serious consequence of unreliability is its effect on peoples' security. The failure of an aeroplane during flight and an explosion or other catastrophe in a nuclear power plant are examples of phenomena, in which the unreliability of a system may not only have economic consequences, but may also lead to the loss of human life.

Reliability, then, is an important factor in the efficiency of a system. The higher the reliability obtained, the more money will be saved, the less time wasted etc. thanks to a reduction in the number of failures and in their consequences. Prevention of system failures as well as control of those failures that do arise, together with the curbing of resultant inconveniences have thus become important problems when striving for efficient management of a system.

12 The background of the study

121 Problem areas in reliability theory

Reliability theory is a science concerned with the laws of occurrence of failures in technical equipment.[1] It deals with the general methods and procedures which must be followed during the planning, preparation, acceptance, transportation, and use of manufactured articles in order to ensure their maximum effectiveness during use, and it developes general methods for evaluating the quality of systems from the known qualities of their component parts.[2] Especially the mathematical theory of reliability, within the framework of which this study belongs, is a body of ideas, mathematical models, and methods directed towards the solution of problems in predicting, estimating, or optimizing the probability of survival, mean life, or, more generally, life distribution of components or systems; other problems considered in reliability theory are those involving the probability of the proper functioning of a system at either a specified or an arbitrary time, or the proportion of time the system is functioning properly.[3]

According to one classification[4], there are six problem areas in reliability theory:

- the criteria and quantitative characteristics of reliability
- the methods of reliability analysis
- the methods of synthesis of complex systems according to criteria of reliability
- the methods of increasing reliability
- the methods of testing equipment for reliability
- the scientific methods of operating equipment with its reliability taken into account

In the first problem area the basic concepts of reliability theory are dealt with: What do we understand by the concept 'reliability'? What are its characteristics? How do we measure

1 POLOVKO (1968), p. XVII
2 GNEDENKO et al. (1969), p. 1
3 BARLOW and PROSCHAN (1965), p. V
4 POLOVKO (1968), p. XIX

and reveal reliability? What is the connection between the indices of economy, efficiency, and reliability? and so on. The quantitative definition of reliability is based on probability theory and the quantitative characteristics of reliability are, in general, expressed as probabilities or expected values concerning the operability/inoperability of the unit in question.

In considering the problems of reliability analysis, we seek methods for calculating and analyzing the characteristics of reliability of a component, a combination of components or of a whole system. Further, the dependence of system reliability on the structure of the system, on the behavioural properties of the components etc. may be derived.

One of the most important problems in connection with the synthesis of complex systems is the development of methods by which reliable systems can be constructed from unreliable elements. Perhaps the best known method of this kind is the redundancy of components; a system of redundant components will function even when some of its components do not.

The reliability of a system can be increased, for example, by introducing redundancy among the strategic components of the system, by decreasing the failure intensity of the system, and by decreasing the repair times. These measures belong to both the design and manufacturing phase of the system as well as its operation and use. An increase in the reliability of a system is, however, always realized on account of its other characteristics, in particular, its weight, size, cost, accuracy etc. From the point of view of economy and efficiency, the increase of reliability is thus beneficial only up to a certain limit.

The vast majority of the problems in developing methods for testing the reliability of equipment are technical rather than mathematical in character. However, test methods and sampling plans of statistical quality control can be used in this reliability testing.

The theoretical basis for preventive maintenance procedures, ratings of standby components, methods of locating failures,

and methods of accumulating and analyzing statistical data on failures of equipment are examples of scientific methods of operating equipment with its reliability taken into account.[1] All of these six problem areas in reliability theory are also problem areas in the mathematical theory of reliability, despite the fact that some of them have a technical rather than a mathematical character. This study is mainly concerned with the first and second problem areas, which clearly come within the scope of the mathematical theory of reliability.

122 On research methods in the mathematical theory of reliability

The mathematical theory of reliability is mainly concerned with probabilities, mean values, probability distributions etc. that refer to the normal operation, failures, repairs etc. of a system, subsystem or component. However, mathematical reliability theory is not only an application of the standard probability theory; reliability problems have a structure of their own and have stimulated the development of new areas in probability theory itself.[2] Renewal theory and the supplementary variable technique, the latter being applied in this study, are good examples of methods which have been intensively developed in the mathematical theory of reliability.

The methods and mathematical models in reliability theory are, in general, either probabilistic or statistical by nature. In this study, statistical problems such as the estimation of component or system life from observed data or testing of hypotheses concerning failure or repair distributions, are not dealt with. The general methodological background of the study thus consists of probabilistic models in reliability theory. The deductions concerning the operation, failure, repair, and, finally, reliability of a system are made from information concerning the probabilistic behaviour of individual components. The behaviour of individual components is assumed to depend on

1 POLOVKO (1968), p. XIX.

2 See BARLOW and PROSCHAN (1965), p.V.

certain known, either general or strictly specified, failure and repair time distributions of these components.

There are several probability tools and methods which are frequently found in reliability literature. Renewal theory, the theory of Markov and semi-Markov processes, birth and death processes, and queuing theory are examples of widely used probability tools in reliability theory. Renewal theory has been applied especially in replacement problems and in general reliability problems of a repairable system. The renewal of a failed component (or of a still operable component) can have various forms. The component can be replaced by an identical new component or it can be subjected to maintenance or repair that completely restores all its original properties. The times of replacements or terminal points of repairs constitute a random flow, the renewal process. Birth and death processes have applications e.g. in redundant systems with repair possibilities. Failure of the redundant component of a system is analogous to the death of an individual in a population, the completion of a repair corresponds to the birth of a new individual, the number of operable components being the size of the population. Methods of queuing theory can be applied to the reliability analysis of systems, where the failure and repair of components correspond to the arrival of customers and service, respectively.

The method adapted in this study is the differential-difference -equation technique, based on Markovian characterization of the system under study. The general class of stochastic processes with discrete states in continuous time that arise in queuing theory, birth-death processes etc., can be characterized as Markov processes provided the full set of random variables needed to specify the state of the process is employed.[1] Now the same is true also for processes in reliability theory.[2] A complete Markovian characterization of the originally non-Markovian system is usually provided, and also in this study,

1 KEILSON and KOOHARIAN (1960), p. 104.

2 See e.g. NATARAJAN (1968), pp. 104-108.

by the inclusion of so-called supplementary variables in the phase of model formulation.[1] The solution of reliability models, which are of the form of partial differential-difference equations, is usually based on Laplace transforms and generating functions(z-transforms). Instead of generating functions, so-called discrete transforms[2] (or binomial moments[3]) have also been commonly used for solving reliability models. The efficiency of discrete transforms has been found to be particularly great in the case of equations with variable coefficients.[4] In this study, the method of Laplace transforms and discrete transforms will be applied for solving the partial differential-difference equations governing the behaviour of the system under study.

123 Special features of reliability problems in a processing plant

One of the most important factors influencing the choice of topic for this study was a research project carried out by the author in the years 1971-72.[5] In this project a simulation model was built up in order to optimize (for certain maintenance activities) the maintenance policy of a typical processing plant, a pulp and paper mill. During this work the

1 The first detailed illustrations of the supplementary variable approach are given in COX (1955) for the case of a queue in equilibrium and in KEILSON and KOOHARIAN (1960) for the case of the general time-dependent version of the queuing problem. Early applications of supplementary variables in reliability problems are e.g. GARG (1963a), GARG (1963b), KULSHRESTHA (1966), KULSHRESTHA (1967), KULSHRESTHA (1968a), KULSHRESTHA (1968b), NATARAJAN (1968) and SRINIVASAN (1968).

2 On discrete transforms see e.g. THIRUVENGADAM and JAISWAL (1964).

3 Discrete transforms are introduced as binomial moments in TAKÁCS (1962), p. 149.

4 Applications of discrete transforms in reliability problems are found e.g. in KULSHRESTHA (1967), NATARAJAN (1968), SRINIVASAN (1968), KULSHRESTHA (1972), NAKAMICHI et al. (1974), KODAMA (1976) and VIRTANEN (1976).

5 For reports on this research see VIRTANEN (1972a) and VIRTANEN (1974a); the project is fully documented in VIRTANEN (1972b).

special features of reliability problems in a processing plant were very clearly revealed. The traditional concepts of reliability were found to be quite insufficient to cover a production system of the type of a processing factory.

A typical processing factory is a system which is composed of a number of production units[1] connected, from the point of view of reliability theory, in series, i.e. all the units in the system must be operable in order to keep the whole system operable. The serial units in the system can be individual components, combinations of components or subsystems. A great many of the units are generally ordinary "operable or inoperable" units: a unit either operates or, after failure, becomes wholly inoperable. Due to the serial coupling of the units, the system is operable or inoperable, respectively.

A processing factory has, however, the special feature, that it usually contains, besides the "normal" operable or inoperable units, also such units which, after failure, instead of becoming completely inoperable, maintain the ability to operate partially (and also to produce partial output). A reduction in the efficiency of the system may arise from different causes, depending e.g. on the type of the failed unit. The failure of a unit which contains only one component may for example give rise to a reduction in efficiency, if the component (and, due to the serial interdependence of the units, the whole system) can operate after the failure with a decreased production rate only. In a processing factory there are usually also stages, where the process, both for operation and for material flow, is branched into parallel channels, where each branch is responsible for a given portion of the capacity of the process.[2] For example, in

1 The term 'unit' is used as an aggregated concept for different levels of elements in the system, i.e. for components, combinations of components, parts of the system (subsystems), or the like, see GNEDENKO et al. (1969), p. 69.

2 This composition must not be confused with redundancy composition (standby or parallel), i.e. with the composition of identical components that does not fail until the last redundant component fails, see PIERUSCHKA (1963), p. 48.

a pulp factory, several, say four, parallel pulp washers may be needed simultaneously in order to handle the whole quantity of pulp flowing in the process. Since the branches operate independently, the failure of one of the branches, instead of making the subsystem (and at the same time the whole system) totally inoperable, allows it to continue in operation, with the intensity of the branches still operating. In the example of the pulp washers given above, the failure of one of the four pulp washers means a reduction in the efficiency of the process amounting to one fourth of the capacity of the process (the pulp washers are assumed to be identical and ordinary operable or inoperable equipment).

As we have seen, failure in the units of a processing plant may have different consequences, depending on the unit in which the failure occurs. The failure may render the whole plant inoperable (in the case of a normal serial unit), or it may have no direct consequence in the operation of the plant (the failure of one of the parallel devices in a redundantly connected unit), or it may make the plant operate with reduced efficiency (the failed unit can be run only at a retarded speed, some of the parallel branches in a unit have failed etc.). The first two cases above are ordinary situations in reliability literature and do not involve any conceptual or definitional difficulties in system reliability. In the last case, however, certain problems arise.

When a production system is in a state, where it is operable with reduced efficiency, it must, if the traditional concepts of reliability are applied, be graded as operable or inoperable (failed). But if this is done, the failure situation is not handled quite correctly. The system is not totally inoperable, and thus also in the sense of efficiency and reliability it must be better than a totally inoperable system. On the other hand the failure has consequences e.g. on the output of the system, whereby the system cannot be as efficient and reliable as a completely undamaged system. Now for a valid determining of reliability, failures having a limiting influence on the operational capability of a system should also be included among the factors decreasing system reliability. They should, however, be included only to the extent to which the efficiency of the system is reduced. For this purpose a more general and more comprehensive concept of reliability is needed.

13 The general problematics of the study and its position in the field of reliability theory

In the preceding section we dealt with the background of the study, both in respect of the theoretical promptings and also in respect of the author's practical experiences in production systems of the processing type, especially the special features of its reliability problems. In making a synthesis of this background we shall in this section propose a general definition and formulation of the problematics of the study. At the same time we shall give a review of the nearest relevant studies in current reliability research.

In the area of the mathematical theory of reliability, reliability analyses of complex systems are quite often based on purely theoretical probabilistic models and concern systems, which do not necessarily have any exact counterpart in the real world. In most cases, however, the analyst has, at least as a mental picture, some real system or class of real systems in mind, to which he may apply his model. In reliability literature, the real world examples of potential applications of systems are most frequently weapon systems, electronic systems or (nuclear) power systems. Reliability problems of production systems of the processing type or, more generally, of systems which have at least the main features of a processing plant, have not been analyzed. When, in addition to the absence of systems of the above mentioned type, the imperfection in the concepts of reliability among the systems with states of reduced efficiency is taken into account, we have entered the problem area of this study and we can set its general definition as follows.

We carry out in this study a conceptual analysis of reliability in order to generalize the intension and to enlarge the extension of reliability in such a way that it also becomes possible to consider the reliability of systems with states of reduced efficiency (i.e. the concepts of system reliability become meaningful in these systems, it becomes possible to measure and reveal this reliability, etc.).

The traditional concepts of reliability (i.e. the concepts defined for two-stage operable or inoperable systems) have already become established in reliability literature. They are presented in the basic textbooks of reliability theory[1] in a completely uniform manner and have subsequently hardly been revalued. The conceptual analysis in the present study is based on these conventional concepts of reliability. An especially good starting point for the analysis is provided by the general mathematical definition of reliability. The two definitions given by Barlow and Proschan[2] and by Gnedenko et al.[3] differ to some extent from each other. We use the latter formulation, and make all our derivations within the framework of this general definition.

In the framework of these new, more comprehensive concepts of reliability we then carry out the reliability analysis of a specific stochastic system. We shall consider the behaviour of the system with the passage of time, and on the basis of that behaviour derive the most important reliability characteristics for the system. Although the system is purely theoretical, it possesses units (subsystems) the operation and behaviour of which is typical of any production system having the property of reduced efficiency. We can even state that the system contains all the principal and most typical elements of a processing plant. One of the units in the system, the subsystem with parallel branches (see subsystem S_4 in section 411), is of a type not considered earlier in reliability literature.

1 See e.g. LLOYD and LIPOW (1962), von ALVEN (1964), BARLOW and PROSCHAN (1965), POLOVKO (1968), GNEDENKO et al.(1969).

2 BARLOW and PROSCHAN (1965), pp. 6-7.

3 GNEDENKO et al. (1969), pp. 74-79.

Systems which, in addition to the states of normal operation ("up-states") and of complete failure ("down-states") have states in which the system is able to operate only partially ("reduced efficiency-states") are quite common in reliability literature. The systems which have earlier been considered are the following:

1 The system contains (in addition to other types of components) just one component such that after failure of this component the whole system begins to operate with reduced efficiency.[1]

2 The system contains a subsystem consisting of two components such that the failure of one or both of these components makes the system operate with reduced efficiency.[2]

3 The system contains a subsystem consisting of several components (of a fixed number $m \geq 2$ or of an unspecified great number) such that the failure of one or more of these components makes the system operate with reduced efficiency.[3]

4 The system contains two subsystems, each one of which has an unlimited number of components, and the failure of any one of these components puts the system into a state of reduced efficiency (the components in the two subsystems are of different types, have different repair priorities etc.).[4]

All the systems mentioned above have common the fact that the amount of reduction in efficiency is assumed to be the same, independently of the type and number of components in the state

1 E.g. GARG (1963a), KULSHRESTHA (1968b), GOVIL and KUMAR (1970), GOVIL (1972).

2 E.g. VARMA (1973), GOVIL (1974).

3 E.g. DAS (1971), GOVIL (1971b) SRIVASTAVA et al. (1971a), SRIVASTAVA et al. (1971c), GUPTA (1973a), GUPTA (1973b), KAPUR and KAPOOR (1975).

4 GOVIL (1970), GOVIL (1971a), SRIVASTAVA et al. (1971b).

of reduced efficiency. For these systems it is not possible, either, that a system already operating with reduced efficiency might further fail in such a way that its efficiency would be further reduced. In practice, however, these assumptions do not seem to be very realistic. It is much more likely that dissimilar components, when they fail, also have dissimilar effects on the efficiency of the system. In general it is also possible that a system already operating with reduced efficiency may meet another failure which will further decrease its efficiency (or make the system completely inoperable). To ensure a better correspondence between the model and reality, the system should also in the model have several possible (reduced) levels of efficiency caused by one or more simultaneous failures.

In the reliability analysis of the systems cited above there also exist some conceptual defects. The behaviour of a system is described by state probabilities. The states for which the probabilities have been derived are either the original elementary states of the system or the three aggregated states, viz. the up-state, the down-state and the state of reduced efficiency. Although the possibility of reduced system efficiency has been taken into account in the definition of the states, this aspect has not, however, been considered in the reliability analysis: conclusions concerning the reliability of the system are always based on the normal operation of the system. Thus, for example, the probability of the up-state at time t is regarded as the availability of the system (at time t). This means that operation with reduced efficiency makes no contribution to availability, although the system operates. There is even a clear error in one of the papers.[1] In that paper, for example, a reliability optimization problem is considered, where the optimization of system reliability (under the steady state) is made according to the number of parallel redundant components in one of the subsystems. The reliability of the system is

1 See GUPTA (1973b), p. 157.

based only on the probability $P_{o,M}$ (the probability of normal operation with all the components in the subsystem operable), although the system is also assumed to be operable with normal efficiency when one of the M components in the subsystem has failed (the corresponding probability is $P_{o,M-1}$). In addition to this there is $P_{o,M-2}$, the probability that the system is operating with reduced efficiency due to the failure of any two components in the subsystem. This is ignored also in the reliability optimization. The more comprehensive consepts of reliability developed in this study allow us, instead, to take such aspects into account.

2 THE OBJECTIVES AND GENERAL STRUCTURE OF THE STUDY

In the preceding chapter, in paragraph 13, we presented the general problematics of this study as a synthesis from the background of the study. In that context we also touched on the objectives and main results of the study, but in such a general form that a more detailed and specified presentation will be necessary. As has already been shown, the objectives of the study are in two directions: the general problem of extending the concepts of reliability and the formulation, solution and use of the reliability model for a particular stochastic system with states of reduced efficiency. Due to the complexity of the system and the general nature of the assumptions regarding the random variables of the reliability model we also attain new results in the development and application of the mathematical methods.

21 The extension of the concepts of reliability

When we consider a system which is not only in one of the two states: up-state (the system is failure-free and thus capable of full performance) or down-state (the system is totally inoperable and under repair), but may also perform its function at one or more levels of reduced efficiency, the conventional concepts of reliability are found to be unsuitable and inadequate: the reliability of the system remains unresolved (there exist situations when the system is neither fully operable nor fully inoperable, so that reliability cannot be determined at all), or it gets a value which contradicts empirical observation (if operation with reduced efficiency is regarded as normal operation, too high reliability is obtained; if a reduction in efficiency is regarded as total inoperability, too low reliability is obtained).

Our first objective is to extend the concepts of reliability in order to make it possible to determine also the reliability of systems with states of reduced efficiency. In making the extension, care must be taken that it is done in a theoretically wellfounded and empirically adequate way. Furthermore there must be no violation of the traditional concepts. These conditions will be fulfilled, when we set for the new concepts the following requirements:

1 Failures having a limiting effect on the efficiency of the system are referred to factors which decrease the reliability of the system, but do this to an extent less than the decrease in reliability caused by a failure resulting in total system inoperability. Further, the degree of reliability decrease is dependent on the degree of reduction in efficiency: the more serious the consequences of the failure, the greater the decrease in reliability of the system.

2 When the new, more comprehensive concepts of reliability are applied to general systems with many levels of performance, we get empirical interpretations analogical to those which result when the conventional concepts of reliability are applied to ordinary two-stage, operable or inoperable systems.

3 When a two-stage, operable or inoperable system is under consideration the new concepts are in agreement with the traditional concepts of reliability.

4 The mathematical definition of the new concepts remains within the limits of the general mathematical definition of reliability.[1]

5 The numerical value of reliability can be determined directly from the behaviour of the system, i.e. from the state probabilities of the system.

This conceptual analysis of reliability will be carried out in Chapter 3. Explicitly we carry out the extension of the concepts only for the quantitative characteristics of reliability.

1 GNEDENKO et al. (1969), pp. 74-79.

We give the general principles of the extension procedure and derive in detail new, more comprehensive reliability characteristics corresponding to the characteristics 'availability', 'reliability' and 'mean time to system failure' of traditional reliability. In the course of the derivation we show that the new characteristics are theoretically well-founded and empirically adequate; the five requirements, and more generally, the objectives laid down for the extension procedure are thereby met.

22 Formulation and use of the reliability model for the reliability analysis of a stochastic system with states of reduced efficiency

The second main objective in the study is to determine and analyze the reliability of a stochastic system, which besides the modes of normal operation and total failure also possesses the property of operation at several different levels of performance (i.e. with reduced efficiency). The system has three operation modes: "normal operation", "operation with reduced efficiency", and "non-operation". We have chosen it as a general representative of systems with states of reduced efficiency. We have tried especially to include in our system the typical main features of a processing factory. In our system these main features have been described by means of the following four types of components:[1]

(i) the ordinary two-stage operable or inoperable components; the failure of any one of the components renders the whole system inoperable (subsystem S_1)

(ii) the functionally multi-stage components, the failure of any one of which makes the component (and the whole system) operate with reduced efficiency; the degree of reduction in efficiency depends on which component has failed (subsystem S_2)

1 A detailed description of the structure of the system is given in section 411.

(iii) the ordinary two-stage operable or inoperable components in parallel redundancy; the system fails only when all the redundant components have failed (subsystem S_3)

(iv) the ordinary two-stage operable or inoperable components in the subsystem formed by independent, parallel branches; the failure of one or more of the components (branches) makes the system operate with reduced efficiency, the degree of reduction in efficiency depending on the number of simultaneous failures among the components (subsystem S_4)

The total inoperability of the subsystem or its operability at a level of reduced performance is a consequence of one or more failures among the components of the subsystem. At any time, the subsystem functioning at the lowest level of performance determines the performance level of the whole system, the subsystems being connected in series. Due to the combinations of the performance levels of the subsystems, the system has a great number of possible levels of performance, ranging from normal operability through different degrees of reduced efficiency to total inoperability.

The system is assumed to be maintained by a single repair facility so that only one failure can be repaired at a time. Because there may be several failures among the components at the same time and there is only one repair facility, the failed components must sometimes queue for repair. In the handling of this queue we assume that the preemptive repeat repair discipline is followed. Under this repair policy, different repair priorities are assigned to different components and different types of failures, and the repairs are carried out according to these priorities.[1]

The components of the system are assumed to fail with constant failure rates, i.e. the failure times are governed by exponent-

1 The preemptive repeat repair discipline in general, and especially in connection with the system of the study, is described in section 412.

ial distributions. The repair times of the components have general distributions, i.e. the repair rates of the components are allowed to be wholly arbitrary functions of time (some regularity conditions must be met, however). Both failure and repair time distributions are peculiar to individual components.[1]

We can now point to the following contributions concerning the structure and properties of the system under study:

1 The system contains a unit (the subsystem with parallel branches) of a type not considered earlier in mathematical reliability literature.

2 The system, consisting of four different types of subsystems with a general number of components in each subsystem, is the largest and most general theoretical system, the reliability of which has been analyzed in the dynamic form.

3 The inclusion in the system of a new type of subsystem and the complexity of the system itself are not only theoretically interesting but also empirically relevant. For all the subsystems there exist clear counterparts in reality among production systems, for example and especially in processing factories.

For the reliability analysis of the system we construct a mathematical model. The formulation of the model starts with the definition of the states for the system.[2] Because the state (at time t) is an exact description of the circumstances prevailing in the system at that time, the behaviour of the system with the passage of time may be found by determining the state probabilities of the system. Due to the general repair time distributions the system is not Markovian. However, by the inclusion of the supplementary variables we provide a complete Markovian

1 Assumptions regarding the random variables are given in section 413.

2 See section 421.

characterization of the system.[1] After the inclusion of the supplementary variables we can set up the model. It gets the form of partial differential - difference equations with variable coefficients.[2] The solution of the model is derived by the application of Laplace transforms and discrete transforms. Both the time-dependent (transient state) and steady state solution are considered.[3] With general repair time distributions, the transient state solution of the model stops at the Laplace transforms of the state probabilities (which, with given repair distributions, we may invert to give the state probabilities). Under the steady state on the other hand, the use of the limit properties of Laplace transforms leads us straight to the state probabilities proper.

In the reliability analysis of the system we link the two main objectives of the study together. The reliability analysis of the system is carried out within the framework of the new extended reliability concepts. The characteristics of this extended reliability are now derived on the basis of the solution of the model, on the basis of the state probabilities, either directly (the generalized availability characteristics) or after some modifications in the original model (the generalized reliability and mean time to system failure characteristics).[4]

23 Development and wider application of the methods

Multi-component repairable systems with general failure and/or repair time distributions are always difficult to handle mathematically. Renewal theory and the Markov process approach with the inclusion of supplementary variables are examples of the

1 The description of the mathematical methods is given in 422.

2 The model is presented in section 423.

3 The derivation of the solution and its results will be found in 424.

4 Paragraph 43 deals with the reliability analysis of the system.

probability tools, with the help of which the reliability analysis of this type of complex system has turned out to be successful. We use the latter approach in this study.

Due to the general repair time distributions in all of the components, the system is not Markovian. But we can characterize the system as a Markov system by employing a set of variables, the supplementary variables, with the help of which a part of the system's history (the time the component under repair has already been being repaired) is included in the state definition of the system.[1] The supplementary variable technique proves to be very efficient also in the complex system under study, in the system of four subsystems with a general number of components in each. The dynamic model for the behaviour of the system can be set up. It gets the form of a set of differential-difference equations with respective boundary and initial conditions. The equations have variable coefficients.

As a consequence of the use of supplementary variables, the equations become partial differential equations in the two time variables. After using the Laplace transforms the equations become algebraic in one variable and remain differential equations in the other variable. The equations thus become easier to solve in the Laplace transforms domain than in the original time domain. At the same time the equations are, however, difference equations in two (state index) variables. The usual technique for solving difference equations is to employ generating functions (z-transforms). But because of the variable coefficients in the equations, the transformed equations would now become partial differential equations also in the transform variables.

1 For a mathematical description of the supplementary variable technique see section 422.

These twice-transformed equations (Laplace transforms and generating functions) with variable coefficients would then not be much easier to solve than the original ones.

Because the use of generating functions in order to solve the model turned out be troublesome or even impossible, we had to find some other way. The method of discrete transforms was the tool that led to the desired result. By using discrete transforms we can transform a discrete set of numbers (or functions) to another discrete set of numbers (or functions).[1] Because in the transforms only multiplying by binomial coefficients and summation are used, the inverse transforms for the discrete transforms are easy to find (whereas the derivation of inverse transforms for Laplace transforms and generating functions may become very problematic). In our model the Laplace-transformed equations, which are differential equations in one variable and difference equations in two variables, become after application of discrete transforms and integration (to get rid of the derivatives) algebraic equations, even linear in the unknown functions. There are not, of course, any difficulties of principle in solving such linear equations. This result, that the discrete transforms lead to usual linear equations, is unknown to reliability literature. In the earlier applications of discrete transforms, different ad hoc -methods have been used for solving the transformed equations.

1 A detailed mathematical description of discrete transforms is given in section 422.

3 THE CONCEPTS OF RELIABILITY IN SYSTEMS WITH STATES OF REDUCED EFFICIENCY

This chapter deals with some problems of definition and interpretation that appear in connection with the content of the concept 'reliability', especially with the quantitative characteristics of the reliability of certain kinds of systems. New definitions for the concepts of reliability, to extend their intension and extension, turn out to be essential in order to make a valid reliability analysis of the system to be presented in the succeeding chapters.

In paragraph 31 we consider the definitions and characteristics of traditional reliability, i.e. reliability that has been defined for two-stage operable or inoperable systems. The paragraph contains a brief summary of the concepts of reliability in the form they usually have in reliability literature. Paragraph 31 also contains a general mathematical definition of reliability which is shown to cover all the quantitative characteristics of reliability: the characteristics can be obtained as particular cases of this general definition.

In paragraph 32 the traditional concepts of reliability are shown to be too narrow in both intension and in extension: the reliability of those systems, which as a consequence of failures have several possible levels of performance, remains on the basis of these definitions unresolved or gets a value that contradicts empirical observation. The concepts of reliability are now extended so that these deficiencies are removed.

31 The traditional concepts of reliability

In the definition of reliability an extremely important role is played by the concept of failure. By failure we mean the occurrence of any condition which renders the system incapable of operating within its specified performance parameter limits.[1] This kind of failure may occur when a device or piece of equipment entirely ceases to operate (e.g. burning out of a light bulb) or when its operation otherwise no longer meets the requirements of the user (e.g. a drop in the voltage of an accumulator). A common factor, however, in traditional reliability considerations is that at any time it must be possible to classify the equipment either as operable (failure-free) or as inoperable (failed). A failure thus either occurs or does not occur, no intermediate position is known (e.g. such that the equipment were only "half-failed").

Definitions of reliability are in general based on the occurrence or non-occurrence of a failure or failures at a given time or during a given time interval. According to their main features the definitions are divided into two groups, qualitative and quantitative definitions. Qualitative definitions concern the general content of the reliability concept on the lines of the definition for failure above. In quantitative definitions the characteristics of reliability are given as probabilities or expected values of a distribution in association with the occurrence of failures.

311 Qualitative definitions of reliability

The following are typical representatives of the group of qualitative definitions of reliability:

1 Reliability is the ability of the equipment to preserve its output characteristics (parameters) within established limits in given conditions of operation.[2]

2 By unit reliability we mean the ability of the unit to maintain its quality under specified conditions of use.[3]

1 von ALVEN (1964), p. 285; very similar definitions for failure are also found e.g. in POLOVKO (1968), p. 9 and in GNEDENKO et al. (1969), p. 71.

2 POLOVKO (1968), p. 1.

3 GNEDENKO et al. (1969), p. 70.

3 Reliability is defined as 'the probability of a successful operation of the device in the manner and under the conditions of intended customer use'.[1]

4 Reliability is the probability of a device performing its purpose adequately for the period of time intended under the operating conditions encountered.[2]

5 Reliability is the probability that a system will perform satisfactorily for at least a given period of time when used under stated conditions.[3]

Although the qualitative definitions of reliability form a group that can be regarded as quite homogeneous, the measurement and rendering of reliability still remain to some extent unresolved. The final specification of the concepts of reliability is made in the form of quantitative definitions.

312 Quantitative characteristics of reliability

The number of quantitative characteristics of reliability is quite large, different indices of reliability playing the determining role when different systems or their different uses are considered. In the following, the three most general and important quantitative characteristics of reliability are presented, viz. reliability,[4] (pointwise) availability and mean time to system failure.

1 LLOYD and LIPOW (1962), p. 20.

2 BARLOW and PROSCHAN (1965), p. 6.

3 von ALVEN (1964), p. 6.

4 The term 'reliability' is commonly used in two different senses in reliability literature. One meaning is the system's general ability to function in the way intended by the user (the definitions in section 311); sometimes the term 'dependability' is used instead of reliability. The other meaning of reliability is one particular characteristic of the concept in question; this characteristic is also called the 'reliability function'. In general, there exists no danger of confusion, the context revealing which of the reliabilities is in question.

The reliability of a system at time t, R(t), is defined as the probability of failure-free operation of the system during the time t[1], i.e.

(3.1) $R(t) = \text{Prob}\left\{\text{the system operates without failure from 0 to t}\right\}$.

Reliability R(t) is typically a characteristic of systems that are non-repairable or are considered as non-repairable. In this context we are only interested in the first failure of the system and the time of its occurrence. The existence of a repair facility, the duration time of the repair etc. do not have any effect on the reliability achieved.

The (pointwise) availability of the system at time t, A(t) is the probability[2]

(3.2) $A(t) = \text{Prob}\left\{\text{the system is operable at time t}\right\}$.

The availability of a system thus depends, not only on the ability of the system to operate without failures, but also on the efficiency with which the repair of the system has been arranged. The availability of an easily failing system may still be high, if the repair times of the system are very short. As a characteristic of reliability A(t) is thus reasonable only in connection with repairable systems.

Mean time to system failure, T, is, as its name shows, the expected value of the time that the system operates uninterruptedly without failing.[3] As far as the characteristic T is concerned, we can consider the time before the first system failure as well as the time from the completion of the repair of a failure to the occurrence of the next failure. Thus T is a characteristic of reliability both for a repairable and for a non-repairable system.

1 This is the definition of reliability (reliability function) given in GNEDENKO et al. (1969), p. 79. Similar definitions of reliability, only slightly differently worded, are given e.g. in BARLOW and PROSCHAN (1965), p. 7 and JORGENSON et al. (1967), p. 14.

2 Cf. BARLOW and PROSCHAN (1965), p. 7 and RAU (1970), p. 239.

3 T is also known by other names, e.g. 'mean time of failure-free operation of the system', GNEDENKO et al. (1969), p. 77 and 'system mean time before failure', RAU (1970), p. 240 etc.

313 The general mathematical definition of reliability

In section 312 the three quantitative characteristics of reliability were defined as probabilities or expected values. In spite of definitional dissimilarities between the different characteristics and thus also of the fact that each of the characteristics gives weight to some extent to different points in the operation of the system, they all measure and reveal the ability of the system to manage the tasks and requirements given to it, i.e. the very reliability that has been set as the object of the measurement. The compatibility of these characteristics is formally revealed in such a way that it turns out to be possible to construct an abstract mathematical concept of reliability which, as particular cases, contains all the most important and usual characteristics of reliability, e.g. the three considered in the foregoing section.[1]

Generally, let the symbol s denote a state, in which a system can exist at a given time. Then $S = \{s\}$ is the set of all the different possible states of the system, called the phase space of the system (two states of the system are considered different only when they differ from each other in some way from the point of view of reliability). The state of the system can be either a scalar or vector quantity. With the passage of time, various changes may take place in the constituent parts of the system; the state of the system changes. Let x(t) denote the state of the system at the instant t. Then $x(t) \in S$ for all values of t ($t \geq 0$). When the stochastic nature of the

1 A definition of reliability in a general form of this kind is presented by BARLOW and PROSCHAN (1965) pp. 6-7 and by GNEDENKO et al. (1969) pp. 74-78; in this study the latter formulation is mainly followed.

state transition is taken into account, the stochastic state (the set of the states) at time t or the state distribution at time t can be described by the random variable X(t). Then each x(t) observed or possible at time t is a particular value of X(t). The ordered set $X = (X(t) \mid t \geq 0)$ of the random variable X(t), indexed to the time variable t, is then a stochastic process describing the course of the states in time. A time series $(x(t) \mid t \geq 0)$ is a realization or trajectory of this stochastic process. It is evident that the value of every trajectory $\hat{x}$ at any time instant $t^* \geq 0$ is one of the elements of the phase space S, i.e. $\hat{x}(t^*) \in S$ for all $\hat{x}$ and for all $t^* \geq 0$. The set of these trajectories we denote by $\hat{X}$.

After definition of the phase space S and the stochastic process X we are able to formulate the general definition of reliability. Let Φ be a functional defined on the trajectories of the process X (on the set $\hat{X}$), whereupon to every trajectory $\hat{x} \in \hat{X}$ there corresponds a unique value $\Phi(\hat{x})$. The reliability φ is now defined as the expected value of that functional:

$$\varphi = E\left\{\Phi(\hat{X})\right\}. \quad (3.3)$$

The final specification of the reliability concept, i.e. the choice of the quantitative characteristics to be used, thus remains dependent on the definition of the functional Φ. In the following we show that e.g. the characteristics considered in section 312 are obtained as particular cases of this general definition of reliability.

Reliability R(t)

Let subset S^o of the phase space S be defined such that the system is inoperable when its state belongs to S^o. Let the functional Φ_1 be defined as follows:

$$\Phi_1(\hat{x}) = \begin{cases} 0 & \text{if, for at least one value } 0 \leq u \leq t,\ \hat{x}(u) \in S^o \\ 1 & \text{otherwise.} \end{cases} \quad (3.4)$$

Then we have

$$\begin{cases} \varphi_1 = E\left\{\Phi_1(\hat{X})\right\} = \text{Prob}\left\{\Phi_1(\hat{X}) = 1\right\} \\ \quad = \text{Prob}\left\{\text{the system does not visit } S^o \text{ before } t\right\} \\ \quad = \text{Prob}\left\{\text{the system operates without failure from } 0 \text{ to } t\right\} \\ \quad = R(t). \end{cases} \quad (3.5)$$

Defining the functional Φ according to equation (3.4) thus leads to the characteristic 'reliability'.

Availability A(t)

Let S^o be the subset of S defined above, and define the functional Φ_2 as follows:

$$\Phi_2(\hat{x}) = \begin{cases} 0 & \text{if } \hat{x}(t) \in S^o \\ 1 & \text{if } \hat{x}(t) \notin S^o. \end{cases} \quad (3.6)$$

Now we have

$$\begin{cases} \varphi_2 = E\left\{\Phi_2(\hat{X})\right\} = \text{Prob}\left\{\Phi_2(\hat{X}) = 1\right\} \\ \quad = \text{Prob}\left\{X(t) \notin S^o\right\} \\ \quad = \text{Prob}\left\{\text{the state of the system at time t does not belong to } S^o\right\} \\ \quad = \text{Prob}\left\{\text{the system is operable at time t}\right\} \\ \quad = A(t), \end{cases} \quad (3.7)$$

and we have achieved the characteristic 'availability'.

Mean time to system failure T

Let the functional Φ_3 be defined by the equation

$$\Phi_3(\hat{x}) = \int_0^\infty I_{S-S^o}(\hat{x}(t))dt, \quad (3.8)$$

where I_{S-S^o} is the indicator function of the set $S-S^o$. i.e.

$$(3.9) \qquad I_{S-S^O}(\hat{x}(t)) = \begin{cases} 1, \text{ if } \hat{x}(t) \in S-S^O \\ 0, \text{ if } \hat{x}(t) \in S^O, \end{cases}$$

and the states in S^O (the inoperable states of the system) are now regarded as absorbing. $\Phi_3(\hat{X})$ thus gives the first time point at which the trajectory $\hat{X}$ falls into the subset S^O. Or in other words, the value of the functional Φ_3 is the lenght of the failure-free operation time of the system. As the expected value for the lenght of this failure-free operation time we obtain the third characteristic of reliability, mean time to system failure T:

$$(3.10) \qquad \varphi_3 = E\{\Phi_3(\hat{X})\} = T.$$

32 Extended concepts of reliability

321 Problems concerning the concepts of reliability in the case of partial reduction in the level of performance of a system

As stated before, the concepts of reliability have taken quite fixed forms in literature on the subject. By the reliability of a system we generally mean the system's ability to fulfil the tasks and requirements given to it, whereas the measurement and revelation of reliability takes place as probabilities or mean values concerning the failure-free operation of the system. In paragraph 31 reliability was considered from the point of view of this traditional approach.

Among production systems there exist, however, several examples which show that the usual concepts of reliability are too narrow to cover all kinds of systems. This is revealed particularly clearly when a production system of the type of a processing plant is considered, as we saw in section 123. The problematic situation comes from the fact that the system is able after failure to operate with reduced efficiency or at a level of reduced performance,[1] instead of becoming wholly inoperable due that failure. We also saw in section 123, that the reduction in the level of performance of the system may be a consequence of the property of an individual component (the component operates, but is e.g. retarded) or of the type of composition of the components (parallel branches in the system, some of the branches having failed).

It is evident that the production efficiency of a system which, as a consequence of failure, has operated at a level of reduced performance cannot be as high as the production efficiency of a similar system which has all the time operated normally. On the other hand, its production efficiency is higher than that of a similar system which, during the time in question, has been totally inoperable. Circumstances like this ought also to be included in the reliability of the system: the reliability of the system should decrease due to the failure, but not, however, as much as in the case of total system failure. But when the traditional concepts of reliability are used, the situation either remains unresolved (the reliability of the system cannot be determined at all), or it is incorrectly treated (the reduction in the level of performance is left unconsidered or the system is regarded as totally inoperable).

In the situation described above the concepts of reliability are closely connected with the reduction of the level of performance of the system and not with the relations between full operability and total inoperability only. In the next section we extend the intension of reliability concepts, especially for the quantitative characteristics, so that the aspects mentioned will be taken into account. The extensions are supported by the empirical interpretation of reliability and they are done without violating traditional concepts: for ordinary two-stage operable or inoperable systems the new extended concepts preserve their previous content and meaning, and the traditional

1 By the level of performance of a production system we mean the actual amount of a system's output per unit of time, cf. HONKO (1974), p. 39.

concepts will be obtained as particular cases of the extended concepts. Further, we remain all the time within the framework of the general mathematical definition of reliability.[1] Finally, we take care that all the requirements set for the new concepts in paragraph 21 are taken into account.

322 Quantitative characteristics of the extended concepts of reliability

3221 Notation and preliminary remarks

In the following we suppose that a system is considered, which has the operational properties described in the previous section. In the real world such a system is typically represented by any processing plant, e.g. a pulp and paper factory. The system has the property that it may not only be either fully operable or totally inoperable but also partially operable. Partial operability of the system may, as indicated above, either originate in the partial operability of a single component of the system, or may be caused by a certain arrangement of parallel components in some subsystem (i.e. branched production lines). It is not therefore necessary to consider the detailed internal structure of a component: the component may be the abstract counterpart of a single device, a group of devices or even a whole plant - i.e., in a special case, it may be a whole (sub)system. The only thing that is essential is that information concerning the behaviour of the component is available for reliability considerations.

Let us assume, that the possible levels of performance of the system are $c_0, c_1, \ldots, c_K$, where $c_0 = 0$ stands for total inoperability, c_K indicates full operability (the level of performance is equal to the capacity C of the system), and $c_1, \ldots, c_{K-1}$ are levels of reduced performance caused by one or more simultaneous failures. The levels of performance $c_0, c_1, \ldots, c_K$ are assumed to be scalar quantities, i.e. possible to be numbered and placed in order. Let the respective proportional levels of performance[1] be denoted by $w_0, w_1, \ldots, w_K$. Then we have $w_0 = 0$, $w_K = 1$, and $0 < w_k < 1$ when $k = 1, 2, \ldots, K-1$.

1 The definition in GNEDENKO et al. (1969), p. 79.

Between the performance levels and the states of the system there is an obvious relation. For each state there is a uniquely determined level of performance; several states, on the other hand, may lead to one and the same performance level. The progression of the states and the levels of performance of the system during a time interval [0, T] is indicated schematically in Figure 3.1, where the state of the system changes at times

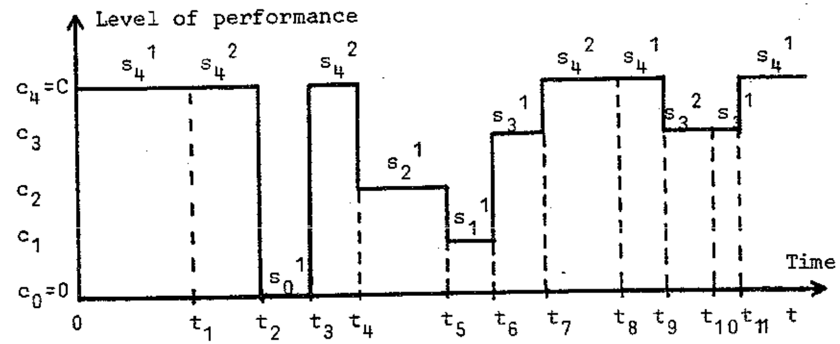

Figure 3.1. The states and levels of performance of a system with the passage of time

$t_1, t_2, \ldots, t_{11}$. Of the eleven state transitions those taking place at times t_1, t_8 and t_{10} are such that they do not indicate any change in the level of performance of the system (e.g. failure of a single component where there are several redundantly connected components still operating, completion of the repair of such a component etc.).

The concepts introduced, 'level of performance' and 'proportional level of performance' do not prevent us from considering

1 Proportional level of performance = ratio of the level of performance and the capacity.

an ordinary operable or inoperable system within this framework. In such a case we only have $K = 1$, when c_0 means the total inoperability of the system and c_1 the total (= capacity level) operability of the system. The only possible proportional levels of performance of the system are $w_0 = 0$ and $w_1 = 1$, respectively.

Let the possible states of the system be numbered as follows: the states corresponding to the performance level c_i ($i = 0,1,\ldots,K$) are $s_i^1, s_i^2,\ldots, s_i^{n_i}$. Let especially s_K^1 denote the state, in which all the components of the system are operating normally. Let the set of these states be denoted by S_i. So we have $S_i = \{s_i^1, s_i^2,\ldots, s_i^{n_i}\}$. The state probabilities of the system we denote by the symbol $P_{s_i^j}(t)$, where

(3.11) $$P_{s_i^j}(t) = \text{Prob}\{\text{the state of the system at time } t \text{ is } s_i^j\},\quad i = 0,1,\ldots,K,\ j = 1,2,\ldots,n_i.$$

By means of this notation not only the current state of the system but also its level of performance is revealed (on the basis of the subscript of the state symbol s_i^j).

3222 Availability as a function of the level of performance

In paragraph 31 the availability $A(t)$ of an ordinary operable or inoperable system was defined as the probability that the system is operable at time t. The reduced performance levels, which are consequences of the failures of certain types of components or subsystems, can be taken into account in the calculation of availability, when a new concept of availability, 'availability of levels of performance', denoted by A_0, is defined as follows[1]

1 In order that the characteristic 'availability of levels of performance' (and the other generalized characteristics to be presented later on in this chapter) would be well-defined, the initial state of the system must be strictly specified. Therefore we assume throughout the work that the system at time t=0 is in the state in which all of its components are operating normally: the initial state of the system is s_K^1. The same assumption has been made, although implicitly, when the traditional concepts of reliability have been defined.

(3.12) $$A_0(c,t) = \text{Prob}\{\text{the level of performance of the system at time } t \text{ is} \geq c\}.$$

While the conventional availability $A(t)$ (at a fixed time t) is simply a number, $A_0(c,t)$ (at time t) is now a function of the level of performance (in the range $0<c \leq C$). We can immediately see that A_0 comes within the framework of the general definition (3.3). Defining, for given values of c and t, the functional Φ as follows

(3.13) $$\Phi_4(\hat{X}) = \begin{cases} 0, & \text{if } \hat{X}(t) \in \bigcup_{c_i<c} S_i \\ 1, & \text{if } \hat{X}(t) \in \bigcup_{c_i\geq c} S_i, \end{cases}$$

we get $A_0(c,t)$ as the expected value of this functional

(3.14) $$\begin{cases} E\{\Phi_4(\hat{X})\} = \text{Prob}\{X(t) \in \bigcup_{c_i\geq c} S_i\} \\ \quad = \text{Prob}\{\text{the level of performance of the system at time } t \text{ is at least } c\} \\ \quad = A_0(c,t). \end{cases}$$

In practice, we can easily calculate $A_0(c,t)$, if the state probabilities of the system are known

(3.15) $$A_0(c,t) = \sum_{c_i\geq c} \sum_{j=1}^{n_i} P_{s_i^j}(t).$$

When calculating the value of $A_0(c,t)$, among the factors decreasing it (from the maximum value 1) are included those failures as a consequence of which the level of performance of the system falls below c. From definition (3.12) and equation (3.15) it can immediately be seen that the "extended availability" A_0, considered as a function of c, takes the aspects presented in section 321 completely into account: also the failures causing only a partial reduction in the level of performance have a decreasing effect on the reliability of the system, but their effect is not, however, as absolute as the effect of failures which result in the complete inoperability of the system.

Further, the ordinary availability A of a two-stage operable or inoperable system remains as a particular case for A_0. For, when we set $c = c_1 = C$ (the capacity of the system; the system has only two possible levels of performance: $c_0 = 0$ and $c_1 = C$) in $A_\theta(c,t)$, we obtain for an operable or inoperable system

$$(3.16)\quad \begin{cases} A_0(C,t) = \sum_{j=1}^{n_1} P_{s_1{}^j}(t) \\ \quad = \text{Prob}\left\{\text{the system is at time } t \text{ in one of the states of full operability}\right\} \\ \quad = \text{Prob}\left\{\text{the system is operable at time } t\right\} \\ \quad = A(t). \end{cases}$$

Just as ordinary availability under the steady state[1] is interpreted as the mean portion of time during which the system is in the functioning state,[2] we also have for $A_0(c,t)$, for all values of $0 < c \leq C$, a completely analogous interpretation under the steady state: it expresses the mean portion of time during which the system is able to operate at least at the level of performance c. From Figure 3.2 we obtain for instance the steady state estimate for $A_0(c)$ as follows (the steady state quantities are denoted without the time variable t)

$$(3.17)\quad \begin{cases} A_0(c) = 1 - \dfrac{(t_2-t_1)+(t_4-t_3)}{T_2 - T_1} = 1 - \dfrac{2+1}{20} = 0.85, \text{ when } c_2 < c \leq C \\ A_0(c) = 1 - \dfrac{t_4-t_3}{T_2-T_1} = 1 - \dfrac{1}{20} = 0.95, \text{ when } c_1 < c \leq c_2 \\ A_0(c) = 1, \text{ when } 0 < c \leq c_1. \end{cases}$$

1 Steady state: the state probabilities of the system and thus, for example, its availability, cease to depend on time.
2 GNEDENKO et al. (1969), p. 112.

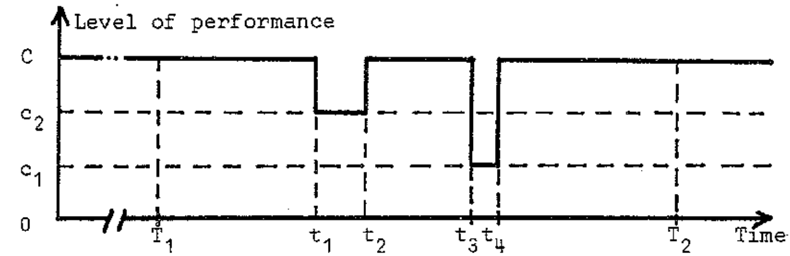

Figure 3.2. Estimation of $A_0(c)$ under the steady state

We have thus shown that extended availability, i.e. availability of levels of performance A_0, meets all the five requirements set out in paragraph 21 for the new extended reliability concepts.

3223 Mean availability of the capacity

When the availability of a traditional operable or inoperable system is considered, the states of the system are divided into two classes only. One class consists of the states of full operability (the level of performance is equal to the capacity) and the other of the states of total inoperability. These classes are denoted $S_1 = \{s_1{}^1, s_1{}^2, \dots, s_1{}^{n_1}\}$ and $S_0 = \{s_0{}^1, s_0{}^2, \dots, s_0{}^{n_0}\}$, respectively (i.e. we have K=1 in the notation of section 3221).

When the system is in one of the states $s_1{}^j \in S_1$, its proportional level of performance is equal to 1, whereas in the states $s_0{}^k \in S_0$ the proportional level of performance is 0. So availability A(t) is also obtained as the probability

$$(3.18)\quad A(t) = \text{Prob}\left\{\text{the proportional level of performance of the system at time } t \text{ is equal to } 1\right\}.$$

Taking the proportional level of performance at time t, denoted by W(t), as a random variable, which can have the value 1 or 0, with probabilities $P_1(t) = \text{Prob}\{W(t)=1\}$ and $P_0(t) = \text{Prob}\{W(t)=0\}$, respectively, we further obtain on the basis of equation (3.18)

$$(3.19)\quad \begin{cases} A(t) = \text{Prob}\left\{W(t) = 1\right\} \\ \quad = \text{Prob}\left\{W(t) = 1\right\} \cdot 1 + \text{Prob}\left\{W(t) = 0\right\} \cdot 0 \\ \quad = E\left\{W(t)\right\}. \end{cases}$$

We find thus that availability $A(t)$ can also be expressed as the mean value of the proportional level of performance at time t.

For a generalized system which has several possible levels of performance $c_0 = 0, c_1, \ldots, c_K = C$ (and several possible proportional levels of performance $w_0 = 0, w_1, \ldots, w_K = 1$, respectively), we can define, analogously to (3.18), an extended availability concept in the form of a distribution. The "availability" of the system is now given by the probabilities of the positive proportional levels of performance[1]

$$(3.20)\quad \begin{cases} P_1(t) = \text{Prob}\left\{W(t) = w_1\right\} \\ P_2(t) = \text{Prob}\left\{W(t) = w_2\right\} \\ \vdots \\ P_K(t) = \text{Prob}\left\{W(t) = w_K\right\}. \end{cases}$$

Taking the relations between the states and the proportional levels of performance into account we can also express the new extended availability or availability function with the help of the state probabilities

$$(3.21)\quad \begin{cases} P_1(t) = \text{Prob}\left\{X(t) \in S_1\right\} = \sum_{j=1}^{n_1} P_{s_1 j}(t) \\ P_2(t) = \text{Prob}\left\{X(t) \in S_2\right\} = \sum_{j=1}^{n_2} P_{s_2 j}(t) \\ \vdots \\ P_K(t) = \text{Prob}\left\{X(t) \in S_K\right\} = \sum_{j=1}^{n_K} P_{s_K j}(t). \end{cases}$$

1 In order to describe the availability of complex systems by more appropriate characteristics than those used for two-stage operable or inoperable systems, J. HRABAK has presented an availability function, which consists of the probabilities of the levels of performance (instead of the proportional levels of performance in equation (3.20)), see HRABAK (1975a) and HRABÁK (1975b).

From the probability distribution (3.20) we can achieve a sigle characteristic when we turn, as in (3.19), to the expected value of the proportional level of performance. This new reliability characteristic we call the 'mean availability of the capacity'[1], and denote it by A_C

$$(3.22)\quad A_C(t) = E\left\{W(t)\right\} = \sum_{i=0}^{K} w_i \,\text{Prob}\left\{W(t) = w_i\right\} = \sum_{i=0}^{K} w_i P_i(t).$$

Also $A_C(t)$ can be calculated with the help of the state probabilities of the system

$$(3.23)\quad A_C(t) = \sum_{i=0}^{K} w_i P_i(t) = \sum_{i=0}^{K} w_i \text{Prob}\left\{X(t) \in S_i\right\} = \sum_{i=0}^{K} \sum_{j=1}^{n_i} w_i P_{s_i j}(t).$$

It is easy to see that the mean availability of the capacity is a reliability characteristic in the sense of the general definition (3.3). In order to prove this we define the functional Φ, for a fixed t, as follows

$$(3.24)\quad \Phi_5(\hat{X}) = \begin{cases} w_0, \text{ if } \hat{X}(t) \in S_0 \\ w_1, \text{ if } \hat{X}(t) \in S_1 \\ \vdots \\ w_K, \text{ if } \hat{X}(t) \in S_K, \end{cases}$$

after which we have

$$(3.25)\quad \begin{cases} E\left\{\Phi_5(\hat{X})\right\} = \sum_{i=0}^{K} w_i \,\text{Prob}\left\{\Phi_5(\hat{X}) = w_i\right\} \\ \quad = \sum_{i=0}^{K} w_i \,\text{Prob}\left\{X(t) \in S_i\right\} \\ \quad = \sum_{i=0}^{K} w_i P_i(t) \\ \quad = A_C(t), \end{cases}$$

1 The term 'mean availability of the capacity' comes from an important property which A_C possesses under the steady state, see considerations later on in this section.

i.e. $A_C(t)$ has been shown to be the expected value of the functional Φ_5' defined on the set $\hat{X}$ of the trajectories of the stochastic process X.

From the definition (3.22) and equation (3.23) we can immediately see that also A_C has a more general intension than the ordinary availability A: also the failures leading to only partial reduction in the level of performance have a decreasing effect on the value of A_C, but this effect is smaller than the effect of failures resulting in total inoperability.

Further, in the arguments for definition (3.21) it became apparent that the availability of a two-stage operable or inoperable system remains as a particular case for the generalized characteristic A_C.

The analogy between the characteristics A and A_C also holds for their properties and interpretations under the steady state, as will be shown in the following.

Under the steady state, the availability of a two-stage operable or inoperable system (denoted by A) gives the mean ratio of the actual system output and the output that would have been produced if there had been no failures during the observation period. In Figure 3.3 the steady state availability A is thus the ratio of the shaded area and the area of the rectangle with the base T_1T_2 and height OC. In Figure 3.4, the same situation is illustrated by means of proportional units on both axes. Time is given as a proportion of the interval T_1T_2, so that 0 corresponds to T_1 and 1 to T_2, and on the vertical axis the proportional level of performance of the system is shown. After this transformation, the steady state availability A is obtained as the shaded area in Figure 3.4.

In reliability considerations, the proportional levels of performance (or the levels of performance or the states) of the system during some interval (e.g. during the interval T_1T_2 in Figure 3.4) are often illustrated in the form of a duration

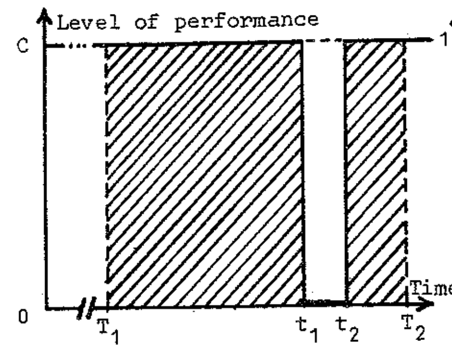

Figure 3.3. Steady state availability as the ratio of the actual and potential outputs of the system

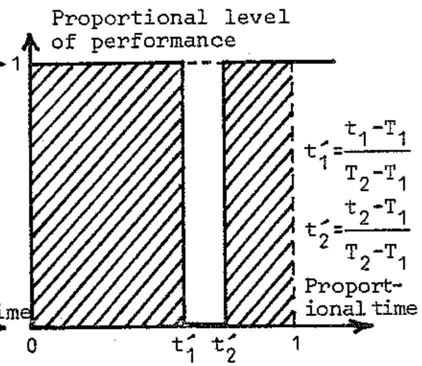

Figure 3.4. Steady state availability as the area under the curve of the proportional level of performance

curve.[1] The duration curve of the proportional level of performance, corresponding to Figure 3.4, is shown in Figure 3.5. From this figure we see, in addition to the area interpretation of Figure 3.4, that the steady state availability expresses the mean portion of time during which the system is in one of the operable states. In Figure 3.5 also the interpretation of availability as the mean proportional level of performance is illustrated.

In Figures 3.3 to 3.5 we have four different steady state properties and interpretations for traditional availability. In the following, A_C, the mean availability of the capacity, is shown to possess completely analogous properties under the steady state as A was shown to possess above.

1 The duration curve of the proportional level of performance: the proportional levels of performance appearing during the interval T_1T_2 are placed in descending order.

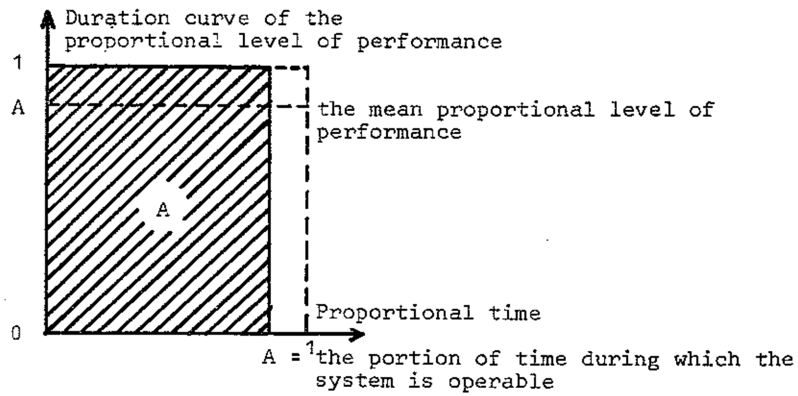

Figure 3.5. Steady state availability as the area under the duration curve of the proportional level of performance, as the mean proportional level of performance and as the portion of time during which the system is operable

First, by analogy with A (cf. Figure 3.3), we note that under the steady state the mean availability of the capacity (=A_C) gives the ratio of the actual system output and the output that would have been possible without any failure during the observation period. This interpretation of A_C is illustrated in Figure 3.6. A_C is the ratio of the shaded area and the area of

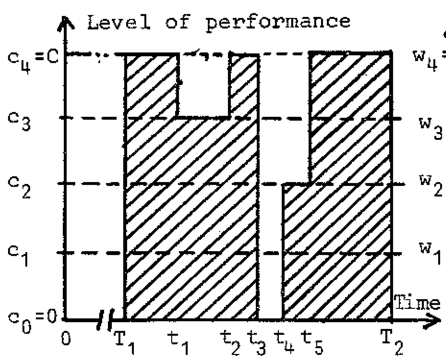

Figure 3.6. A_C as the ratio of the true and potential outputs of the system

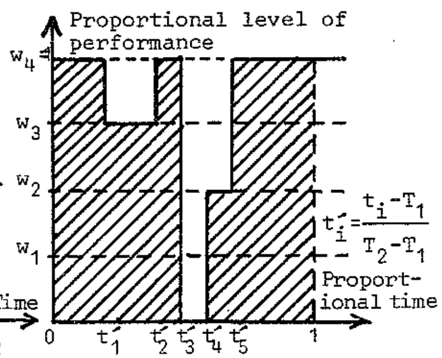

Figure 3.7. A_C as the area under the curve of the proportional level of performance

the whole rectangle with the base T_1T_2 and height 0C. In Figure 3.7 both time and performance quantities are expressed in proportional units. In this case A_C is the shaded area under the proportional level of performance-curve (cf. Figure 3.4).

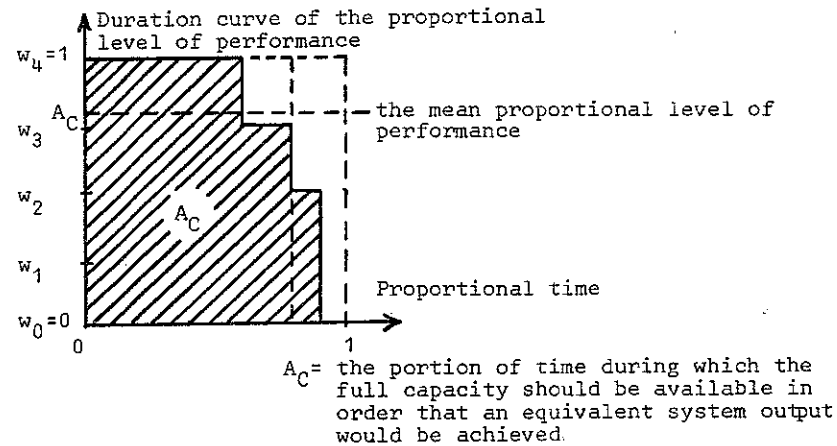

Figure 3.8. A_C as the area under the duration curve of the proportional level of performance, as the mean proportional level of performance and as the portion of time for equivalent full capacity operation

Figure 3.8 shows the duration curve of proportional level of performance corresponding to Figure 3.7. From this figure we can see, by analogy with Figure 3.5, three different properties and interpretations for A_C: A_C is the mean proportional level of performance of the system (the definition of the characteristic A_C), A_C expresses the area under the duration curve of the proportional level of performance (a derived property for A_C), and A_C equals the portion of time during which the full capacity of the system should be available in order to achieve an equivalent system output (it is on the basis of this interpretation of A_C, that the term 'mean availability of the capacity' has been chosen).

The extended availability concepts A_0 and A_C clearly emphasize different points in the operational ability of the system. A_0

measures the reliability of the system from the point of view of momentary operational ability (e.g. the ability of a power system to maintain different levels in the production of power). A_C, on the other hand, measures the reliability of the system from the point of view of production output (e.g. the amount of energy produced by the power system, relative to the capacity of the system).

3224 Reliability as a function of the level of performance

When an ordinary two-stage operable or inoperable system is considered, and it is important from the point of view of the tasks and use of the system that it should remain operable during the whole of an interval 0 to t, the most appropriate characteristic for reliability is usually the reliability or reliability function R(t).

The system is observed during the interval [0,t] and it is noted whether the system fails or not during that interval; eventual repair of the system and recommencement of operation are not considered. The reliability of a two-stage operable or inoperable system is then the probability that the system will not fail during the interval [0,t].

When, however, a system with many possible levels of performance is considered, also failures causing only a partial decrease in the level of performance must be taken into account in reliability calculations. This requirement is fulfilled when a new, generalized reliability concept, 'reliability of levels of performance', denoted by R_0, is introduced

(3.26) $$R_0(c,t) = \text{Prob}\left\{\text{the level of performance of the system during the interval } [0,t] \text{ is at least } c\right\}.$$

As with A_0 before, R_0 is a function both of time (for $t \geq 0$) and of level of performance (for $0 < c \leq C$), i.e. with fixed t, $R_0(c,t)$ is a function of c, whereas the ordinary reliability R(t) is simply a number.

Defining, for given values of c and t, the functional Φ in (3.3) as follows

(3.27) $$\Phi_6(\hat{x}) = \begin{cases} 0, & \text{if there exist at least one } u,\ 0<u\leq t, \text{ such that } \hat{x}(u) \in \bigcup_{c_i<c} S_i \\ 1, & \text{otherwise,} \end{cases}$$

we have

(3.28) $$\begin{aligned} E\left\{\Phi_6(\hat{X})\right\} &= \text{Prob}\left\{\Phi_6(\hat{X}) = 1\right\} \\ &= \text{Prob}\left\{X(v) \in \bigcup_{c_i\geq c} S_i \text{ for all } v,\ 0\leq v\leq t\right\} \\ &= \text{Prob}\left\{\text{the level of activity of the system during the interval } [0,t] \text{ is at least } c\right\} \\ &= R_0(c,t), \end{aligned}$$

i.e. $R_0(c,t)$ has been shown to be the expected value of a functional defined on the set $\hat{X}$ of the trajectories of the stochastic process X.

In order to calculate a particular value $R_0(c,t)$ of the reliability of levels of performance directly from the state probabilities, we must partly redefine the states of the system. Those states in which the level of performance of the system amounts at least to c remain unchanged. Similarly, the classification of these states into subsets S_i does not change. Those states, however, in which the level of performance is less than c, are aggregated and defined as absorbing states. When the system gets into an absorbing state it is held to be unreliable with regard to the level c of performance. Let the set of absorbing states be denoted by S_c. Then we have

(3.29) $$S_c = S_0 \cup S_1 \cup \ldots \cup S_r,$$

where r is determined from the condition

(3.30) $$c_r < c,\ c_{r+1} \geq c.$$

Now the value of $R_0(c,t)$ is obtained as the sum of the (original) state probabilities

$$(3.31)\qquad R_0(c,t) = \sum_{i=r+1}^{K} \sum_{j=1}^{n_i} P_{s_i^j}(t)$$

or as the complement probability of the aggregated absorbing states of the system

$$(3.32)\qquad R_0(c,t) = 1 - \sum_{i=0}^{r} \sum_{j=1}^{n_i} P_{s_i^j}(t).$$

Obviously the reliability of levels of performance, considered also as a function of c, is a generalization of ordinary reliability, and thus for a two-stage operable or inoperable system we can achieve reliability R as a particular case of R_0 by setting c = C (the capacity of the system) in the expression of $R_0(c,t)$, i.e.

$$(3.33)\qquad R(t) = R_0(C,t)$$

for a two-stage operable or inoperable system.

We have obtained R_0 as a generalization of R in a similar manner as we obtained A_0 as a generalization of A. Such a generalization of R, that would be analogous to the generalization A_C of availability A, has no relevant empirical interpretation, and is therefore not presented.

3225 Mean time to system failure as a function of the level of performance

It is very important for certain (two-stage operable or inoperable) systems that their uninterrupted operation time is as long as possible. For such systems the mean time to system failure T is usually the most natural and relevant characteristic for reliability. In order to get an analogical reliability characteristic for a system with many possible levels of performance, we introduce a new generalized characteristic 'mean operation time of levels of performance before failure'. For fixed c ($0 < c \leq C$), we define

(3.34) $T_0(c)$ = the mean value of that time period after which the level of performance of the system for the first time becomes less than c.

We can immediately note that also $T_0(c)$ can be obtained from the general definition (3.3). Therefore we choose the functional Φ in (3.3) as follows

$$(3.35)\qquad \Phi_7(\hat{X}) = \int_0^{\infty} I_{S-S_c}(\hat{X}(t))dt$$

where I_{S-S_c} is the indicator function of the set $S-S_c$ (cf. equation (3.9)), and S_c is the set of the absorbing states defined in (3.29). The functional Φ_7 thus expresses the time at which the level of performance of the system for the first time drops below the level c. Then we have

$$(3.36)\qquad E\left\{\Phi_7(\hat{X})\right\} = T_0(c),$$

i.e. we have expressed $T_0(c)$ as the expected value of the functional Φ_7 as is presupposed by (3.3).

For a two-stage operable or inoperable system which has only one class of operable states (the level of performance corresponding to the capacity), the ordinary mean time to system failure T is obtained as a particular case of T_0[1]

(3.37)
$T_0(C)$ = mean value for that time period after which the level of performance of the system for the first time becomes less than the capacity
= mean time to the first system failure = T.

1 In equations (3.16), (3.33) and (3.37) we have shown that the traditional characteristics A, R and T in the case of a two-stage operable or inoperable system can be obtained as particular values of the generalized characteristics A_0, R_0 and T_0, respectively. Because the operability of the system in this case always means operability at "full capacity" level of performance, we could (footnote continues)

In practice, the easiest way to find the value of the mean time to system failure T is usually the integration of the reliability function[1]

(3.38) $$T = \int_0^\infty R(t)\, dt.$$

The generalized characteristic T_0 is obtained in the same way, as the integral of R_0[2]

(3.39) $$T_0(c) = \int_0^\infty R_0(c,t)\, dt.$$

The extending of the concepts of reliability to cover also systems with many possible levels of performance has been made above in detail for three quantitative characteristics of reliability. The characteristics availability, reliability and mean time to system failure have got, as for their generalizations, the more comprehensive characteristics availability of levels of performance, mean availability of the capacity, reliability of levels of performance and mean operation time of levels of performance before failure. On the basis of these examples and the other argumentation it is easy to make a corresponding generalization for any other characteristic of reliability. If the principles given earlier are followed it is possible to ensure that in making such a generalization, a natural connection with empirical interpretation as well as with the tradition of reliability theory will be preserved.

(cont.) also have written the equations in the following form

(3.16)' $A_0(c,t) = A_0(C,t) = A(t), \quad 0 < c \leq C$

(3.33)' $R_0(c,t) = R_0(C,t) = R(t), \quad 0 < c \leq C$

(3.37)' $T_0(c) = T_0(C) = T, \quad 0 < c \leq C.$

We have chosen the particular value c=C, however, because it is more meaningful in content. The importance of the equations (3.16)', (3.33)' and (3.37)' is that they show the new characteristics to be in full agreement with the traditional ones of an ordinary two-stage system.

1 von ALVEN (1964), pp. 235-236; for (3.38) to hold, the integral on the right hand side of (3.38) must exist.

2 cf. footnote 1.

4 RELIABILITY ANALYSIS OF A STOCHASTIC SYSTEM WITH STATES OF REDUCED EFFICIENCY: A MODEL BASED ON THE SUPPLEMENTARY VARIABLE TECHNIQUE AND DISCRETE TRANSFORMS

41 Description of the system under study

411 General structure of the system

In general, an operating system can be classified as one of two types, according to whether it is continuously or only intermittently used. In this study a system of the first type is dealt with. The system to be considered is composed of four types of independent components, i.e. of four independent subsystems, designated S_1, S_2, S_3 and S_4.

Subsystem S_1 consists of all the ordinary operable or inoperable components of the system which from the point of view of reliability theory are connected in series.[1] Thus S_1 is able to operate as long as all of its components are operable and fails when any one of its components fails. The number of components in S_1 is K (K≥1). The components are independent and generally non-identical.

Subsystem S_2 consists of L (L≥1) components, generally non-identical, and the system has the property that the failure of one component merely reduces the efficiency of the subsystem (and the whole system) instead of making it completely inoperable.[2] Subsystem S_2 therefore either operates normally or, after a failure, operates at a level of reduced performance. It is further assumed that there is not more than one failure at a time in S_2.

1 In the real system on the other hand, the components need not physically be one after another: the components may be situated anywhere in the system.

2 What has been stated earlier of the physical location of the components of S_1, holds also for S_2.

Subsystem S_3 contains M (M≥2) identical components which are redundantly connected. Thus all M components must fail for subsystem S_3 to fail. Redundancy in S_3 is assumed to be in parallel, i.e. all the components start operating together as soon as the system is put into operation and S_3 fails when the last component fails. The components operate and fail independently of each other.

Subsystem S_4 is composed of N (N≥2) identical independent components which are so connected that each one of them covers one Nth part of the capacity of subsystem S_4. Thus the failure of one component in S_4 does not make the subsystem completely inoperable but only reduces its level of performance by an amount 1/N of the full capacity. Only when all N components have failed does subsystem S_4 become totally inoperable.

In order to operate normally the system must have all of its subsystems operable, subsystems S_1, S_2, S_3 and S_4 being in series. In the case of one or more component failures the level of performance of the system is the minimum of the levels of performance of the subsystems. The structure of the system is shown in the form of a reliability diagram in Figure 4.1.

Summing up the logical structure of the system we can state that

1 the system operates normally (with the level of performance attaining the capacity of the system), when
(a) all the components of the system are in the state of normal operation or
(b) all the components in subsystems S_1, S_2 and S_4 operate normally and 1 to M-1 components in S_3 have failed (but at least one component in S_3 operates normally)

2 the system operates at a level of reduced performance, when all the components in S_1 and at least one component in S_3 are operable and

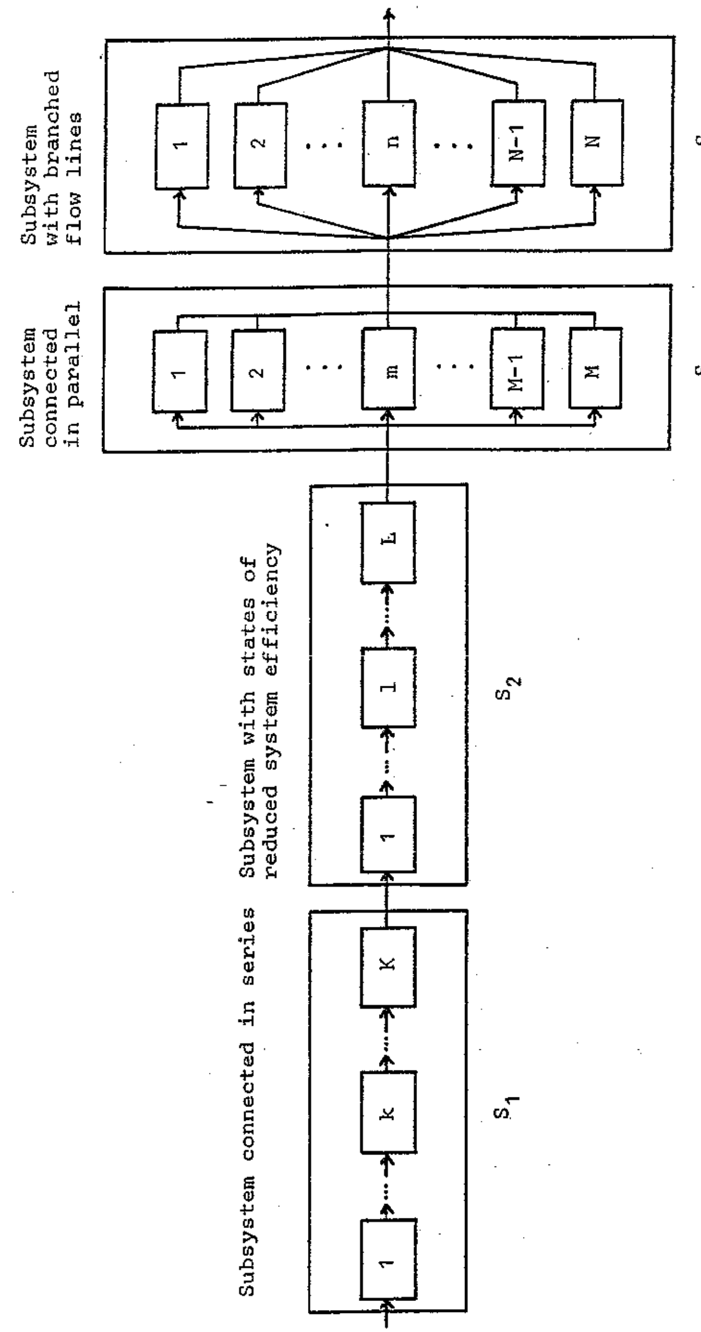

Figure 4.1. Reliability diagram of the system

(a) one of the components in S_2 has failed, leading to a reduction in the level of performance in S_2, and all N components in S_4 are operable (the level of performance of the system is that of subsystem S_2) or
(b) the components in S_2 operate normally but n ($1 \leq n \leq N-1$) of the N components in S_4 have failed (the level of performance of the system is the fraction (N-n)/N of the capacity of the system) or
(c) one of the components in S_2 has failed, reducing the efficiency of S_2, and n ($1 \leq n \leq N-1$) of the N components in S_4 have failed (the performance level of the system is the minimum of the levels of performance of subsystems S_2 and S_4)

3 the system is completely inoperable, when
(a) one of the components of S_1 has failed, or
(b) all M components of S_3 have failed, or
(c) all N components of S_4 have failed.

Further it is assumed that in a case of system failure (cases 3a to 3c above) the additional failure of any other component is not possible (the system is not operating and cannot fail more, either). The system must be repaired and put into operation again before a new failure can arise. In the case of a failure with a level of reduced performance, it is, however, possible for any of the still operating components to fail (excluding the components of S_2 in cases 2a and 2c above). A failure in this situation simply means that the level of performance of the system remains reduced, is possibly reduced still more, or the system becomes completely inoperable.

412 The repair policy for the system

The system is assumed to be maintained by a single repair facility so that only one failure can be repaired at a time. Thus, if a functioning component fails before completion of the repair of a component which has failed earlier, one or other of these components must wait as if in a queue until the repair of the other component is completed.

In the case of a system failure due to the failure of a component in subsystem S_1, the whole system is inoperable until the failed component in S_1 has been repaired. The repair of a component in subsystem S_2, however, does not interrupt the operation of the system, subsystem S_2 (and the whole system) continuing to operate at a level of reduced performance during the repair of S_2. In S_3, the failed components are not repaired until all M components have failed. In the case of this system failure all the failed components in S_3 are repaired and the system remains inoperable during the repair. In S_4 failed components are repaired one component at a time. The operation of the system can continue in spite of the repair work (with the exception of the case of a system failure due to the total failure of subsystem S_4).

Because there may be several components in the failed state at the same time and there is only one repair facility, the failed components must queue for repair. In the handling of the queue different repair policies may be introduced. In this study we consider the preemptive repeat repair discipline.[1] Under the preemptive repeat repair policy different repair priorities are assigned to different types of failures. The three classes of failures, given in order from the highest to the lowest priority, are the following:

1 Class C_1: system failures. A failure of class C_1 makes the system completely inoperable so that it is important to get the failure repaired as soon as possible. There are three possibilities for a failure of class C_1:
(a) one of the components in S_1 fails
(b) the last operable component of S_3 fails
(c) the last operable component of S_4 fails.

2 Class C_2: failures leading to a decrease in the level of performance of the system. A failure of class C_2 occurs when

1 For different policies in system repair see GOVIL and KUMAR (1970).

(a) one of the components in S_2 fails or

(b) a component in S_4 (if it is not the last operable component in S_4, see 1c above) fails.

Within class C_2 it is assumed that repair of the S_2-components has priority over the S_4-components, there being two subclasses in C_2.

3 Class C_3: failures with no direct effect on the operation of the system. A failure of class C_3 arises when a component in S_3 (not the last one, however) fails, the system maintaining its level of performance unchanged.

Under the preemptive repeat repair policy the queue discipline is as follows:

1 components with a failure of the highest priority class are repaired first

2 the occurrence of a failure of a higher priority class immediately terminates the repair of a component with a failure of a lower priority class and repair of the component with the failure of the higher priority class begins

3 repair of a component which has been removed from the repair facility due to the priority given to the repair of a component in a higher priority class must be started again from scratch, i.e. repair work already done, but left unfinished, is not taken into account.

413 Stochastic elements in the system

The system is a stochastic system because the failure times and repair times are stochastic by nature. The failure and repair times are assumed to be random variables with known distributions. The following assumptions are made concerning the components of the system and the distributions of the random variables.

The components of S_1 are assumed to operate independently and generally to be different. The failure times in S_1 are assumed to be exponentially distributed with parameters $\eta_1, \eta_2, \dots, \eta_K$, respectively. This means in other words that the components in S_1 have constant failure rates, viz. $\eta_1, \eta_2, \dots, \eta_K$. The repair times of the components in S_1 have general distributions, i.e. the repair rates of the components are arbitrary functions of time.[1] If the repair rates of the components are denoted by $\alpha_k(t)$, k=1,2,...,K, then the density functions of the repair times become[2]

$$(4.1) \qquad a_k(t) = \alpha_k(t)\, e^{-\int_0^t \alpha_k(t)dt}, \quad k=1,2,\dots,K,\ t\geq 0.^3$$

Like the components of S_1, also the components of S_2 are assumed to be independent and not necessarily identical. The constant failure rates of the components are denoted by λ_l, l=1,2, ...,L, and the arbitrary repair rate functions by $\beta_l(t)$, l=1,2, ...,L. The failure times and repair times of the components of S_2 are thus governed by exponential and general distributions, respectively. The density functions of the general repair time distributions are of the form

$$(4.2) \qquad b_l(t) = \beta_l(t)\, e^{-\int_0^t \beta_l(t)dt}, \quad l=1,2,\dots,L,\ t\geq 0.$$

The components in subsystem S_3 behave independently of each other (and independently of the components of the other subsystems), but they are identical. They thus have a common failure time distribution which is assumed to be exponential; the

1 General distribution: the type of the distribution has not been specified, i.e. it may be exponential, gamma, Weibull, normal etc. Some regularity conditions concerning the rate function must be fulfilled, however: the rate function must be integrable over the whole non-negative t-axis.

2 For relations between rate and density functions see e.g. POLOVKO (1968), pp. 36-37.

3 Only non-negative random variables are considered: repair times cannot be negative.

failure rates of the components are constant (= μ). According to the general repair policy of the system, subsystem S_3 will be not repaired until all M components have failed and in this case all the components are repaired together. Thus the repair times and their distribution are not peculiar to the components but to the whole subsystem. The repair times of S_3 are assumed to obey a general distribution, the repair rate being $\gamma(t)$. The probability density of the repair time distribution is thus

$$(4.3) \qquad c(t) = \gamma(t)\, e^{-\int_0^t \gamma(t)dt}, \quad t\geq 0.$$

As in S_3 the components are also in S_4 identical and independent. The components are assumed to have a constant failure rate ν and an arbitrary repair rate $\varphi(t)$, both common to all the components. The density function of the repair time distribution of a component is

$$(4.4) \qquad f(t) = \varphi(t)\, e^{-\int_0^t \varphi(t)dt}, \quad t\geq 0.$$

Table 4.1 shows the assumptions made concerning the components of the system. The properties of the components are itemized in the table by subsystems.

Subsystem	Number of components	Components identical/ non-identical	Failure rate of a component	Probability density of failure time distribution	Repair rate of a component	Probability density of the repair time distribution
S_1	K	Non-id.	η_k	$\eta_k\, e^{-\eta_k t}$	$\alpha_k(t)$	$a_k(t)=\alpha_k(t)e^{-\int_0^t \alpha(t)dt}$
S_2	L	Non-id.	λ_1	$\lambda_1\, e^{-\lambda_1 t}$	$\beta_1(t)$	$b_1(t)=\beta_1(t)e^{-\int_0^t \beta_1(t)dt}$
S_3	M	Id.	μ	$\mu\, e^{-\mu t}$	$\gamma(t)^{x)}$	$c(t)=\gamma(t)e^{-\int_0^t \gamma(t)dt}$
S_4	N	Id.	ν	$\nu\, e^{-\nu t}$	$\varphi(t)$	$f(t)=\varphi(t)e^{-\int_0^t \varphi(t)dt}$

x) repair rate for all the components in S_3 together

Table 4.1. Failure time and repair time distributions of the components of the system

42 Formulation and solution of the model

421 The states of the system

In section 411 we made the assumption that all the components of the system behave independently of each other. This assumption means at the same time that also the four subsystems behave independently of each other. Due to the single repair facility, it is not possible, however, to base the formulation of the model on four separate models (one model for each subsystem); the system must be considered as a whole.

We shall describe the behaviour of the system with the passage of time by means of mutually exclusive and totally exhaustive probabilities, i.e. the state probabilities. The first step, therefore, in the formulation of our model is to define the possible states for the system. In a reliability model the state of the system is an exact description of whether the system, the subsystems and the components are capable of operating or not and whether the repair facility is busy or not (and where). We find that the system has 14 qualitatively different states, in one of which the system must be at any instant t ($t \geq 0$). The states of the system and their characteristic features are presented in Table 4.2.

State	S	S_1	S_2	S_3	S_4	Symbol of the state probability	Remarks on the possible values of k, l, m and n			
							k	l	m	n
1	O	O	O	O_m	O_0	$P_{0,0,m,0}$			0,...,M-1	
2	R	O	O	O_m	M_n	$P_{0,0,m,n}$			0,...,M-1	1,...,N-1
3	R	O	M_l	O_m	O_0	$P_{0,l,m,0}$		1,...,L	0,...,M-1	
4	R	O	M_l	O_m	O_n	$P_{0,l,m,n}$		1,...,L	0,...,M-1	1,...,N-1
5	F	O	O	O_m	F	$P_{0,0,m,N}$			0,...,M-1	
6	F	O	O	F	O_0	$P_{0,0,M,0}$				
7	F	O	O	F	O_n	$P_{0,0,M,n}$				1,...,N-1
8	F	O	R_l	O_m	F	$P_{0,l,m,N}$		1,...,L	0,...,M-1	
9	F	O	R_l	F	O_0	$P_{0,l,M,0}$		1,...,L		
10	F	O	R_l	F	O_n	$P_{0,l,M,n}$		1,...,L		1,...,N-1
11	F	F_k	O	O_m	O_0	$P_{k,0,m,0}$	1,...,K		0,...,M-1	
12	F	F_k	O	O_m	O_n	$P_{k,0,m,n}$	1,...,K		0,...,M-1	1,...,N-1
13	F	F_k	R_l	O_m	O_0	$P_{k,l,m,0}$	1,...,K	1,...,L	0,...,M-1	
14	F	F_k	R_l	O_m	O_n	$P_{k,l,m,n}$	1,...,K	1,...,L	0,...,M-1	1,...,N-1

Table 4.2. The states of the system

In Table 4.2 we have used the following symbols

1 for the whole system S

- O: system is operating with normal efficiency
- R: system is operating with reduced efficiency and is under repair
- F: system is in the failed state and under repair

2 for subsystem S_1

- O : S_1 is operable (none of its components has failed)
- F_k: S_1 is inoperable due to the failure of its kth component, the failed component is undergoing repair (k=1,2,...,K)

3 for subsystem S_2

- O : S_2 is operable with normal efficiency
- R_l: S_2 is operable with reduced efficiency (the lth component of S_2 has failed); the failed component is not being repaired (l=1,2,...,L)
- M_l : S_2 is operable with reduced efficiency and its failed component no. l is undergoing repair (l=1,2,...,L)

4 for subsystem S_3

- O_m: S_3 is operable, m of its components have failed (m=0,1,...,M-1)
- F : S_3 is in the failed state (all M components have failed), the whole subsystem is undergoing repair

5 for subsystem S_4

- O_0: S_4 is operable with normal efficiency (none of its components has failed)
- O_n: S_4 is operable with reduced efficiency (efficiency is reduced by an amount n/N of the normal efficiency)
- M_n: S_4 is operating with reduced efficiency, one of the n failed components is under repair
- F : S_4 is in the failed state, one of the N failed components is under repair.

From Table 4.2 and from the list of symbols we can see that out of the 14 states in Table 4.2 only state 6 is a single state, i.e. the conditions prevailing among the subsystems and components are uniquely specified. All the other states in Table 4.2 are certain kinds of aggregated superstates which further have a number of substates. For instance, state 1 has M substates,

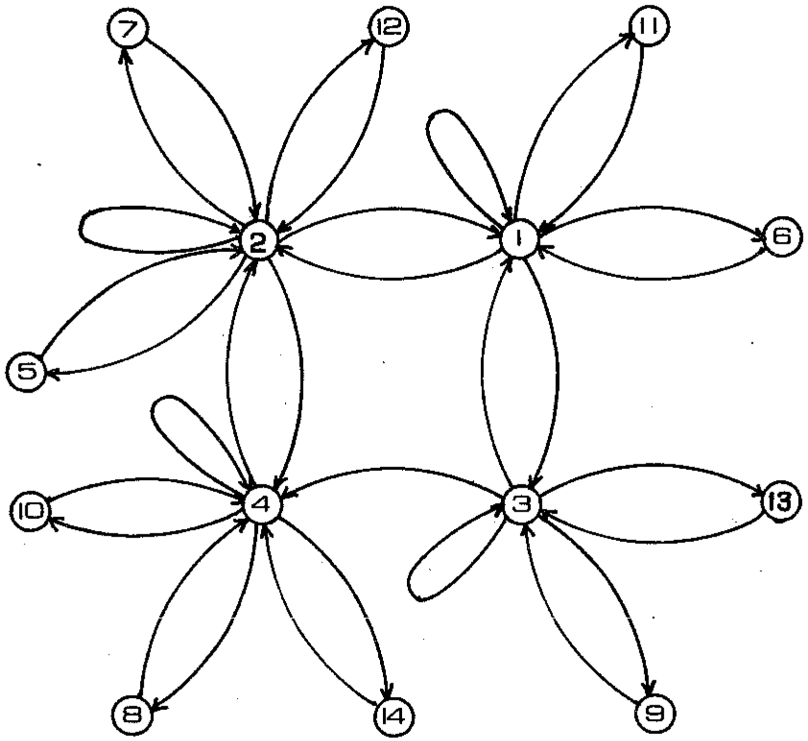

Figure 4.2. State transition diagram of the system

one substate for every number of failed components of S_3. But, from the point of view of reliability theory, these M substates are qualitatively equal and can so be aggregated as a superstate. The same holds for the other 12 superstates in Table 4.2.

Figure 4.2 describes the possible one-step transitions between the states, from 1 to 14. As will be seen, transitions within states 1, 2, 3 and 4 (between the substates of these superstates) are also possible. For example, a transition takes place from state 2

1 to state 1, when there is only one failed component in S_4 (n=1), and the repair of this component has been

completed,

2 to state 4, when one of the components in S_2 fails and preempts in the repair facility (the repair of the component in S_4 is interrupted),

3 to state 5, when there is only one component operable in S_4 (n=N-1) and this last component fails before completion of the repair of the component in S_4 that is undergoing repair,

4 to state 7, when there is only one component operable in S_3 (m=M-1) and this last component fails, resulting in a system failure; repair is begun immediately on subsystem S_3,

5 to state 12, when one of the components in S_1 fails, resulting in a system failure, the failed S_1-component preempts in the repair facility,

6 to the state 2 itself (i.e. another substate of state 2), when one of the still operating components of S_4 fails (if it is not the last), or when the repair of one of the failed components of S_4 is completed (and the repair of another failed S_4-component begins), or when one of the operable components in S_3 fails (if it is not the last).

422 Mathematical description of the methods used in the model

The failure times of the components of the system are assumed to be exponentially distributed, whereas all the repair time distributions are allowed to be general, having an arbitrary repair rate. Because of these general repair time distributions the behaviour of the system can not be described by any Markov process. The future behaviour of the system depends, as well as on the present state of the system and on the transition probabilities between the states, also on one part of the history of the system, viz., on the repair time that has elapsed for the component undergoing repair. The awkwardness of the system's not being Markovian has been overcome by the supplementary variable technique.[1] With this technique, the analysis depends crucially on the inclusion of the supplementary variable x which denotes the time that has elapsed during which a component has been undergoing repair. By introducing the variable x (or in other words by including the repair time in the state definition) the process describing the behaviour of the system becomes Markovian and all the equations for the behaviour of the system can be formulated using unconditional one-step transition probabilities. For example, instead of using in the formulation of the equations the state probability $P_{0,0,m,n}(t)$, $m=0,1,\dots,M-1$, $n=1,2,\dots,N-1$, where

(4.5) $P_{0,0,m,n}(t)$ = probability that at time t all the components in S_1 and S_2 are operable and m of the M components in S_3 and n of the N components in S_4 have failed (the failure of the S_4-components resulting in a reduction in the level of performance of the system), and one of the failed components in S_4 is undergoing repair,

we introduce the joint probability density functions $p_{0,0,m,n}(x,t)$, $m=0,1,\dots,M-1$, $n=1,2,\dots,N-1$. The probabilistic interpretation of the function $p_{0,0,m,n}(x,t)$ is as follows

(4.6) $p_{0,0,m,n}(x,t)dx$ = probability that at time t all the components in S_1 and S_2 are operable and m of the M components in S_3 and n of the N components in S_4 have failed (the failure of the S_4-components resulting in a reduction in the level of performance of the system), and one of the failed components in S_4 is undergoing repair so that the time that has elapsed since this repair started lies between x and x+dx.

1 A detailed illustration of the supplementary variable technique is given in COX (1955), see also KEILSON and KOOHARIAN (1960).

The variable x appearing in the term $p_{0,0,m,n}(x,t)$ is precisely the supplementary variable with the help of which a part of the system's history (the repair time that has elapsed) has been included. It is clear that we get the state probability $P_{0,0,m,n}(t)$ from the probability density $p_{0,0,m,n}(x,t)$ by integration[1]

$$(4.7)\qquad P_{0,0,m,n}(t) = \int_0^\infty p_{0,0,m,n}(x,t)dx,\quad m=0,1,\dots,M-1,\quad n=1,2,\dots,N-1.$$

The use of the supplementary variable x in association with states 3 to 14 is analogous to the case of state 2 considered above. However, in the case of state 1 there is no need for the use of the supplementary variable x, because the system is not under repair when this state is visited.

Due to the supplementary variables used, the equations governing the behaviour of the system, i.e. the state equations, become partial differential equations because there are now two time variables, the original variable t (time from the instant when the system was for the first time put into operation) and the supplementary variable x. In order to get rid of the other derivatives (those calculated with respect to t) we apply the method of Laplace transforms. We denote the Laplace transform of a function f(t) by $\bar{f}(s)$, i.e.

$$(4.8)\qquad L\left\{f(t)\right\} = \bar{f}(s) = \int_0^\infty e^{-st} f(t)dt,\quad Re(s)>0.$$

With the help of this transform all the state equations are transformed into the Laplace transform domain, where the subsequent derivations will be made. The results are obtained as Laplace transforms of the ordinary solutions, but with given values of the system's parameters these expressions may be in-

1 We have, of course, $p_{0,0,m,n}(x,t)=0$, when $x>t$.

verted to give the final results. The inverse transforms are obtained by using the well known Bromwich's integral formula[1]

$$(4.9) \qquad L^{-1}\left\{\bar{f}(s)\right\} = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\,\bar{f}(s)ds, \quad t > 0.$$

The asymptotic or steady state values for the quantities of the system may also be obtained without the use of formula (4.9), by using the final-value theorem[2]

$$(4.10) \qquad \lim_{t\to\infty} f(t) = \lim_{s\to 0} s\bar{f}(s).$$

As the system is a continuous time, discrete state system, the state equations become differential-difference equations: partial differential equations with respect to t and x in the time domain (with respect to s in the Laplace transform domain) and difference equations with respect to m and n (number of failed components in subsystems S_3 and S_4, respectively). Furthermore, the state equations are differential-difference equations with variable coefficients which means that the usual generating function technique for solving these equations leads to new partial differential equations which are very difficult to solve. It is in order to solve such types of equations that discrete transforms have been developed. Discrete transforms[3] can be characterized in the uni-indexed case as follows. Let $g_0(s)$, $g_1(s),\ldots,g_{N-1}(s)$ be a set of non-negative functions. The discrete transforms $h_1(s)$, $h_2(s),\ldots,h_N(s)$ of these functions are defined as the expressions

$$(4.11) \qquad h_m(s) = \sum_{k=m}^{N}\binom{k}{m}g_{N-k}(s) = \sum_{k=0}^{N-m}\binom{N-k}{m}g_k(s), \quad m=1,2,\ldots,N.$$

Then the functions $g_k(s)$, $k=0,1,\ldots,N-1$ may be obtained as the inverse transforms of the functions $h_m(s)$,[4] $m=1,2,\ldots,N$

1 SPIEGEL (1965), p. 201.

2 SPIEGEL (1965), p. 20.

3 See THIRUVENGADAM and JAISWAL (1964); discrete transforms are also known as binomial moments, see TAKÁCS (1962), p. 149.

4 See KULSHRESTHA (1972), p. 146.

$$(4.12) \qquad \begin{cases} g_k(s) = \displaystyle\sum_{m=0}^{k}(-1)^m\binom{N-k+m}{m}h_{N-k+m}(s) \\ \quad\;\;\; = \displaystyle\sum_{m=N-k}^{N}(-1)^{m-N+k}\binom{m}{N-k}h_m(s), \qquad k=0,1,\ldots,N-1. \end{cases}$$

In the bi-indexed case discrete transforms are defined analogously.[1] The discrete transforms of the non-negative functions $g_{m,n}(s)$, $m=0,1,\ldots,M-1$, $n=0,1,\ldots,N-1$, are

$$(4.13) \qquad \begin{cases} h_{j,k}(s) = \displaystyle\sum_{m=j}^{M}\binom{m}{j}\sum_{n=k}^{N}\binom{n}{k}g_{M-m,N-n}(s) \\ \quad\;\;\; = \displaystyle\sum_{m=0}^{M-j}\binom{M-m}{j}\sum_{n=0}^{N-k}\binom{N-n}{k}g_{m,n}(s), \qquad j=1,2,\ldots,M, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k=1,2,\ldots,N. \end{cases}$$

Again, we can obtain the original functions $g_{m,n}(s)$, $m=0,1,\ldots,M-1$, $n=0,1,\ldots,N-1$, in closed form as the following inverse transforms[2]

$$(4.14) \qquad \begin{cases} g_{m,n}(s) = \displaystyle\sum_{j=0}^{m}(-1)^j\binom{M-m+j}{j}\sum_{k=0}^{n}(-1)^k\binom{N-n+k}{k}h_{M-m+j,N-n+k}(s) \\ \quad\;\;\; = \displaystyle\sum_{j=M-m}^{M}(-1)^{j-M+m}\binom{j}{M-m}\sum_{k=N-n}^{N}(-1)^{k-N+n}\binom{k}{N-n}h_{j,k}(s), \\ \quad\;\;\; m=0,1,\ldots,M-1, \; n=0,1,\ldots,N-1. \end{cases}$$

By applying the discrete transforms (4.11) and the double discrete transforms (4.13) to the set of differential-difference equations, the latter are transformed into differential equations which are no longer difference equations. By integration the equations may be derived in a form where they are ordinary algebraic equations with respect to certain unknown functions of s. These equations form a linear system and are thus easy to solve. By using the inverse transforms (4.12) and (4.14), it is possible to obtain the original unknown functions of s in the differential-difference equations.

1 See NATARAJAN (1968), p. 109.

2 NATARAJAN (1968), p.110.

423 Formulation of the model

4231 Notation

First we shall define the following random variables in order to specify the state of the system explicitly. Let, at time $t \geq 0$, measured from the start of the system,

$$K(t) = \begin{cases} 0, & \text{if all the components in } S_1 \text{ are operable,} \\ k, & \text{if the kth component of } S_1 \text{ is in the failed state,} \\ & k = 1,2,\dots,K; \end{cases}$$

$$L(t) = \begin{cases} 0, & \text{if all the components in } S_2 \text{ are operating normally,} \\ l, & \text{if the lth component of } S_2 \text{ is in the state of} \\ & \text{reduced efficiency, } l = 1,2,\dots,L; \end{cases}$$

$M(t)$ = the number of failed components in S_3;

$N(t)$ = the number of failed components in S_4;

$U_k(t)$ = the elapsed repair time of the kth component of S_1 under repair, $k = 1,2,\dots,K$;

$V_l(t)$ = the elapsed repair time of the lth component of S_2 under repair, $l = 1,2,\dots,L$;

$W(t)$ = the elapsed repair time of the whole subsystem S_3 under repair;

$Y(t)$ = the elapsed repair time of the S_4- component under repair

With the help of these random variables we can now define the state probability and joint probability density functions to be used in the formulation of the model, i.e. in the formulation of the state equations of the system.

Thus we define (see also Table 4.2 and equations (4.5) and (4.6))

(4.15) $$P_{k,l,m,n}(t) = \text{Prob}\left\{K(t)=k, L(t)=l, M(t)=m, N(t)=n\right\},$$
$$k=0,\dots,K,\ l=0,\dots,L,\ m=0,\dots,M,\ n=0,\dots,N,$$ [1]

1 On the basis of the assumption made in section 411 (two subsystems cannot be in the failed state simultaneously) we have
$P_{k,l,M,n}(t) = 0$ for all $l=0,\dots,L$ and $n=0,\dots,N-1$, when $k > 0$,
$P_{k,l,m,N}(t) = 0$ for all $l=0,\dots,L$ and $m=0,\dots,M-1$, when $k > 0$,
$P_{k,l,M,N}(t) = 0$ for all $k=0,\dots,K$ and $l=0,\dots,L$.
These state probability functions do not appear in the model, either.

(4.16) $$P_{0,0,m,n}(x,t)dx = \text{Prob}\left\{K(t)=0, L(t)=0, M(t)=m, N(t)=n, x \leq Y(t) \leq x+dx\right\},\ m=0,\dots,M-1,\ n=1,\dots,N,$$

(4.17) $$P_{0,l,m,n}(x,t)dx = \text{Prob}\left\{K(t)=0, L(t)=l, M(t)=m, N(t)=n, x \leq V_l(t) \leq x+dx\right\},\ l=1,\dots,L,\ m=0,\dots,M-1,\ n=0,\dots,N-1$$

(4.18) $$P_{0,l,M,n}(x,t)dx = \text{Prob}\left\{K(t)=0, L(t)=l, M(t)=M, N(t)=n, x \leq W(t) \leq x+dx\right\},\ l=0,\dots,L,\ n=0,\dots,N-1,$$

(4.19) $$P_{0,l,m,N}(x,t)dx = \text{Prob}\left\{K(t)=0, L(t)=l, M(t)=m, N(t)=N, x \leq Y(t) \leq x+dx\right\},\ l=1,\dots,L,\ m=0,\dots,M-1,$$

(4.20) $$P_{k,l,m,n}(x,t)dx = \text{Prob}\left\{K(t)=k, L(t)=l, M(t)=m, N(t)=n, x \leq U_k(t) \leq x+dx\right\},\ k=1,\dots,K,\ l=0,\dots,L,\ m=0,\dots,M-1,\ n=0,\dots,N-1.$$

The symbols to be used in connection with the failure and repair time distributions of the components were already given in Table 4.1 and are not repeated here. We introduce the following additional notation, however

(4.21) $$\sum_{k=1}^{K} \eta_k = \eta,$$

(4.22) $$\sum_{l=1}^{L} \lambda_l = \lambda.$$

All the other notations will be explained in the context of their first appearance.

4232 Differential-difference equations governing the behaviour of the system: the state equations

In order to formulate the mathematical model for the system, let us consider the behaviour of the system in the neighbourhood of an arbitrary time t ($t > 0$). Let us especially consider the behaviour of the system between two times t and $t+\Delta$ so close to each other that during that interval the system can only hold its state or make just one one-step transition into an adjacent state. By connecting the state of the system at time $t+\Delta$ to those at time t from which the state is attainable by the occurrence or non-occurrence of failures and the completion of repair in the interval Δ, we get, for example for state 2, to the first order terms in Δ, the following forward difference equation[1]

$$
(4.23)\quad \begin{cases} p_{0,0,m,n}(x+\Delta,t+\Delta)dx = p_{0,0,m,n}(x,t)dx \times \\ \left\{ \prod_{k=1}^{K} (1-\eta_k\Delta) \prod_{l=1}^{L} (1-\lambda_l\Delta)(1-\mu\Delta)^{M-m}(1-\nu\Delta)^{N-n}[1-\varphi(x)\Delta] \right\} \\ + (1-\delta_{m,0})p_{0,0,m-1,n}(x,t)dx\ (M-m+1)\mu\Delta \\ + (1-\delta_{n,1})p_{0,0,m,n-1}(x,t)dx\ (N-n+1)\nu\Delta, \\ m=0,1,\dots,M-1,\ n=1,2,\dots,N-1, \end{cases}
$$

where

$$
(4.24)\quad \delta_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}
$$

is the Kronecker delta.

Letting $\Delta \to 0$, equation (4.23) results in the following partial differential equation (with respect to the time variables x and t) which is also a difference equation with respect to m and n, the number of failed components in subsystems S_3 and S_4, respectively[2]

$$
(4.25)\quad \begin{cases} [\partial/\partial x + \partial/\partial t + \eta + \lambda + (M-m)\mu + (N-n)\nu + \varphi(x)]p_{0,0,m,n}(x,t) \\ = (1-\delta_{m,0})\mu(M-m+1)p_{0,0,m-1,n}(x,t) \\ + (1-\delta_{n,1})\nu(N-n+1)p_{0,0,m,n-1}(x,t), \\ m=0,1,\dots,M-1,\ n=1,2,\dots,N-1. \end{cases}
$$

1 On the use of supplementary variable technique in the derivation of differential-difference equations governing the behaviour of a stochastic system see e.g. KEILSON and KOOHARIAN (1960), pp. 104-106, KULSHRESTHA (1968a), pp. 161 - 162, KULSHRESTHA (1968b), pp. 233-235 and VIRTANEN (1974b), pp. 34-35.

2 For achieving (4.25), some conditions for continuity must be fulfilled, see LINDELÖF (1932), pp. 17-19 and KEILSON and KOOHARIAN (1960), p. 105.

As above, by connecting the various state probabilities at times t and t + Δ and letting $\Delta \to 0$, we have the following set of differential-difference equations governing the behaviour of the system

$$
(4.26)\quad \begin{cases} [d/dt + \eta + \lambda + (M-m)\mu + N\nu]P_{0,0,m,0}(t) \\ = (1-\delta_{m,0})\mu(M-m+1)P_{0,0,m-1,0}(t) + \delta_{m,0}\int_0^\infty P_{0,0,M,0}(x,t)\gamma(x)dx \\ + \sum_{k=1}^{K} \int_0^\infty P_{k,0,m,0}(x,t)\alpha_k(x)dx \\ + \sum_{l=1}^{L} \int_0^\infty P_{0,l,m,0}(x,t)\beta_l(x)dx \\ + \int_0^\infty P_{0,0,m,1}(x,t)\varphi(x)dx, \quad m=0,1,\dots,M-1, \end{cases}
$$

$$
(4.27)\quad \begin{cases} [\partial/\partial x + \partial/\partial t + \eta + \lambda + (M-m)\mu + (N-n)\nu + \varphi(x)]p_{0,0,m,n}(x,t) \\ = (1-\delta_{m,0})\mu(M-m+1)p_{0,0,m-1,n}(x,t) \\ + (1-\delta_{n,1})\nu(N-n+1)p_{0,0,m,n-1}(x,t), \\ m=0,1,\dots,M-1,\ n=1,2,\dots,N-1, \end{cases}
$$

$$
(4.28)\quad \begin{cases} [\partial/\partial x + \partial/\partial t + \eta + (M-m)\mu + (N-n)\nu + \beta_l(x)]p_{0,l,m,n}(x,t) \\ = (1-\delta_{m,0})\mu(M-m+1)p_{0,l,m-1,n}(x,t) \\ + (1-\delta_{n,0})\nu(N-n+1)p_{0,l,m,n-1}(x,t), \\ l=1,2,\dots,L,\ m=0,1,\dots,M-1,\ n=0,1,\dots,N-1, \end{cases}
$$

$$
(4.29)\quad \begin{cases} [\partial/\partial x + \partial/\partial t + \varphi(x)]p_{0,0,m,N}(x,t) = \nu p_{0,0,m,N-1}(x,t), \\ m=0,1,\dots,M-1, \end{cases}
$$

$$
(4.30)\quad [\partial/\partial x + \partial/\partial t + \gamma(x)]p_{0,0,M,n}(x,t) = 0, \quad n=0,1,\dots,N-1,
$$

$$
(4.31)\quad [\partial/\partial x + \partial/\partial t + \varphi(x)]p_{0,l,m,N}(x,t) = 0, \quad l=1,2,\dots,L,\ m=0,1,\dots,M-1,
$$

$$
(4.32)\quad [\partial/\partial x + \partial/\partial t + \gamma(x)]p_{0,l,M,n}(x,t) = 0, \quad l=1,2,\dots,L,\ n=0,1,\dots,N-1,
$$

$$(4.33)\quad \begin{cases} [\partial/\partial x + \partial/\partial t + \alpha_k(x)]p_{k,0,m,n}(x,t) = 0, \\ k=1,2,\ldots,K,\ m=0,1,\ldots,M-1,\ n=0,1,\ldots,N-1, \end{cases}$$

$$(4.34)\quad \begin{cases} [\partial/\partial x + \partial/\partial t + \alpha_k(x)]p_{k,l,m,n}(x,t) = 0, \\ k=1,2,\ldots,K,\ l=1,2,\ldots,L,\ m=0,1,\ldots,M-1,\ n=0,1,\ldots,N-1. \end{cases}$$

Consideration of repair completions and the occurrence of failures with higher priority when a component with lower priority failure is under repair, leads to the following boundary conditions[1]

$$(4.35)\quad \begin{cases} p_{0,0,m,n}(0,t) = \sum_{k=1}^{K} \int_0^\infty p_{k,0,m,n}(x,t)\alpha_k(x)dx \\ + \sum_{l=1}^{L} \int_0^\infty p_{0,l,m,n}(x,t)\,\beta_l(x)dx + \delta_{m,0}\int_0^\infty p_{0,0,M,n}(x,t)\gamma(x)dx \\ + \int_0^\infty p_{0,0,m,n+1}(x,t)\varphi(x)dx + \delta_{n,1}N\nu P_{0,0,m,0}(t), \\ m=0,1,\ldots,M-1,\ n=1,2,\ldots,N-1, \end{cases}$$

$$(4.36)\quad \begin{cases} p_{0,l,m,n}(0,t) = \sum_{k=1}^{K} \int_0^\infty p_{k,l,m,n}(x,t)\alpha_k(x)dx \\ + \delta_{m,0}\int_0^\infty p_{0,l,M,n}(x,t)\gamma(x)dx + \delta_{n,N-1}\int_0^\infty p_{0,l,m,N}(x,t)\varphi(x)dx \\ + \delta_{n,0}\lambda_l P_{0,0,m,0}(t) + (1-\delta_{n,0})\lambda_l \int_0^\infty p_{0,0,m,n}(x,t)dx, \\ l=1,2,\ldots,L,\ m=0,1,\ldots,M-1,\ n=0,1,\ldots,N-1, \end{cases}$$

$$(4.37)\quad p_{0,0,m,N}(0,t) = 0, \quad m=0,1,\ldots,M-1,$$

$$(4.38)\quad \begin{cases} p_{0,0,M,n}(0,t) = \delta_{n,0}\mu P_{0,0,M-1,0}(t) \\ + (1-\delta_{n,0})\mu \int_0^\infty p_{0,0,M-1,n}(x,t)dx, \quad n=0,1,\ldots,N-1, \end{cases}$$

1 A detailed derivation of boundary conditions in the context of using supplementary variables can be found in KEILSON and KOOHARIAN (1960), p. 106.

$$(4.39)\quad p_{0,l,m,N}(0,t) = \nu_0\int_0^\infty p_{0,l,m,N-1}(x,t)dx, \quad l=1,2,\ldots,L,\ m=0,1,\ldots,M-1,$$

$$(4.40)\quad p_{0,l,M,n}(0,t) = \mu_0\int_0^\infty p_{0,l,M-1,n}(x,t)dx, \quad l=1,2,\ldots,L,\ n=0,1,\ldots,N-1,$$

$$(4.41)\quad \begin{cases} p_{k,0,m,n}(0,t) = \delta_{n,0}\eta_k P_{0,0,m,0}(t) \\ + (1-\delta_{n,0})\eta_k \int_0^\infty p_{0,0,m,n}(x,t)dx, \\ k=1,2,\ldots,K,\ m=0,1,\ldots,M-1,\ n=0,1,\ldots,N-1, \end{cases}$$

$$(4.42)\quad \begin{cases} p_{k,l,m,n}(0,t) = \eta_k \int_0^\infty p_{0,l,m,n}(x,t)dx, \\ k=1,2,\ldots,K,\ l=1,2,\ldots,L,\ m=0,1,\ldots,M-1,\ n=0,1,\ldots,N-1. \end{cases}$$

If we assume that the system is without failure at the outset, we get the following initial conditions

$$(4.43)\quad P_{0,0,m,0}(0) = \delta_{m,0}, \quad m=0,1,\ldots,M-1,$$

$$(4.44)\quad p_{k,l,m,n}(x,0) = 0,\ k=0,1,\ldots,K,\ l=0,1,\ldots,L,\ m=0,1,\ldots,M,\ n=0,1,\ldots,N.$$

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4241 Laplace transforms of the state equations

In equations (4.27) - (4.34) there are partial derivatives both with respect to variable x and variable t. The first step in solving the model is to get rid of one of these derivatives. This can be done by taking the Laplace transforms (with respect to variable t) of the equations (4.26) - (4.42). After taking the Laplace transforms, in the (x,s) - domain there appear derivatives only with respect to x. Applying the Laplace transform and using the notation (4.8), the equations (4.26) - (4.42) with the initial conditions (4.43) and (4.44) become

$$\left\{\begin{aligned}&[s + \eta + \lambda + (M-m)\mu + N\nu]\overline{P}_{0,0,m,0}(s) = \delta_{m,0} + \\ &(1-\delta_{m,0})\mu(M-m+1)\overline{P}_{0,0,m-1,0}(s) + \delta_{m,0}\int_0^\infty \overline{p}_{0,0,M,0}(x,s)\gamma(x)dx \\ &+ \sum_{k=1}^{K}\int_0^\infty \overline{p}_{k,0,m,0}(x,s)\alpha_k(x)dx + \sum_{l=1}^{L}\int_0^\infty \overline{p}_{0,l,m,0}(x,s)\beta_l(x)dx \\ &+ \int_0^\infty \overline{p}_{0,0,m,1}(x,s)\varphi(x)dx, \quad m=0,1,\ldots,M-1,\end{aligned}\right. \tag{4.45}$$

$$\left\{\begin{aligned}&[\partial/\partial x + s + \eta + \lambda + (M-m)\mu + (N-n)\nu + \varphi(x)]\overline{p}_{0,0,m,n}(x,s) \\ &= (1-\delta_{m,0})\mu(M-m+1)\overline{p}_{0,0,m-1,n}(x,s) \\ &+ (1-\delta_{n,1})\nu(N-n+1)\overline{p}_{0,0,m,n-1}(x,s), \\ &m=0,1,\ldots,M-1,\ n=1,2,\ldots,N-1,\end{aligned}\right. \tag{4.46}$$

$$\left\{\begin{aligned}&[\partial/\partial x + s + \eta + (M-m)\mu + (N-n)\nu + \beta_l(x)]\overline{p}_{0,l,m,n}(x,s) \\ &= (1-\delta_{m,0})\mu(M-m+1)\overline{p}_{0,l,m-1,n}(x,s) \\ &+ (1-\delta_{n,0})\nu(N-n+1)\overline{p}_{0,l,m,n-1}(x,s), \\ &l=1,2,\ldots,L,\ m=0,1,\ldots,M-1,\ n=0,1,\ldots,N-1,\end{aligned}\right. \tag{4.47}$$

$$\left\{\begin{aligned}&[\partial/\partial x + s + \varphi(x)]\overline{p}_{0,0,m,N}(x,s) = \nu\overline{p}_{0,0,m,N-1}(x,s), \\ &m=0,1,\ldots,M-1,\end{aligned}\right. \tag{4.48}$$

$$[\partial/\partial x + s + \gamma(x)]\overline{p}_{0,0,M,n}(x,s) = 0, \quad n=0,1,\ldots,N-1, \tag{4.49}$$

$$[\partial/\partial x + s + \varphi(x)]\overline{p}_{0,l,m,N}(x,s) = 0, \quad l=1,2,\ldots,L,\ m=0,1,\ldots,M-1, \tag{4.50}$$

$$[\partial/\partial x + s + \gamma(x)]\overline{p}_{0,l,M,n}(x,s) = 0, \quad l=1,2,\ldots,L,\ n=0,1,\ldots,N-1, \tag{4.51}$$

$$\left\{\begin{aligned}&[\partial/\partial x + s + \alpha_k(x)]\overline{p}_{k,0,m,n}(x,s) = 0, \\ &k=1,2,\ldots,K,\ m=0,1,\ldots,M-1,\ n=0,1,\ldots,N-1,\end{aligned}\right. \tag{4.52}$$

$$\left\{\begin{aligned}&[\partial/\partial x + s + \alpha_k(x)]\overline{p}_{k,l,m,n}(x,s) = 0, \\ &k=1,2,\ldots,K,\ l=1,2,\ldots,L,\ m=0,1,\ldots,M-1,\ n=0,1,\ldots,N-1,\end{aligned}\right. \tag{4.53}$$

$$\left\{\begin{aligned}&\overline{p}_{0,0,m,n}(0,s) = \sum_{k=1}^{K}\int_0^\infty \overline{p}_{k,0,m,n}(x,s)\alpha_k(x)dx \\ &+ \sum_{l=1}^{L}\int_0^\infty \overline{p}_{0,l,m,n}(x,s)\beta_l(x)dx + \delta_{m,0}\int_0^\infty \overline{p}_{0,0,M,n}(x,s)\gamma(x)dx \\ &+ \int_0^\infty \overline{p}_{0,0,m,n+1}(x,s)\varphi(x)dx + \delta_{n,1}N\nu\overline{P}_{0,0,m,0}(s), \\ &m=0,1,\ldots,M-1,\ n=1,2,\ldots,N-1,\end{aligned}\right. \tag{4.54}$$

$$\left\{\begin{aligned}&\overline{p}_{0,l,m,n}(0,s) = \sum_{k=1}^{K}\int_0^\infty \overline{p}_{k,l,m,n}(x,s)\alpha_k(x)dx \\ &+ \delta_{m,0}\int_0^\infty \overline{p}_{0,l,M,n}(x,s)\gamma(x)dx + \delta_{n,N-1}\int_0^\infty \overline{p}_{0,l,m,N}(x,s)\varphi(x)dx \\ &+ \delta_{n,0}\lambda_l\overline{P}_{0,0,m,0}(s) + (1-\delta_{n,0})\lambda_l\int_0^\infty \overline{p}_{0,0,m,n}(x,s)dx, \\ &l=1,2,\ldots,L,\ m=0,1,\ldots,M-1,\ n=0,1,\ldots,N-1,\end{aligned}\right. \tag{4.55}$$

$$\overline{p}_{0,0,m,N}(0,s) = 0, \quad m=0,1,\ldots,M-1, \tag{4.56}$$

$$\left\{\begin{aligned}&\overline{p}_{0,0,M,n}(0,s) = \delta_{n,0}\mu\overline{P}_{0,0,M-1,0}(s) \\ &+ (1-\delta_{n,0})\mu\int_0^\infty \overline{p}_{0,0,M-1,n}(x,s)dx, \quad n=0,1,\ldots,N-1,\end{aligned}\right. \tag{4.57}$$

$$\overline{p}_{0,l,m,N}(0,s) = \nu\int_0^\infty \overline{p}_{0,l,m,N-1}(x,s)dx, \quad l=1,2,\ldots,L,\ m=0,1,\ldots,M-1 \tag{4.58}$$

$$\overline{p}_{0,l,M,n}(0,s) = \mu\int_0^\infty \overline{p}_{0,l,M-1,n}(x,s)dx, \quad l=1,2,\ldots,L,\ n=0,1,\ldots,N-1, \tag{4.59}$$

$$\left\{\begin{aligned}&\overline{p}_{k,0,m,n}(0,s) = \delta_{n,0}\eta_k\overline{P}_{0,0,m,0}(s) \\ &+ (1-\delta_{n,0})\eta_k\int_0^\infty \overline{p}_{0,0,m,n}(x,s)dx, \\ &k=1,2,\ldots,K,\ m=0,1,\ldots,M-1,\ n=0,1,\ldots,N-1,\end{aligned}\right. \tag{4.60}$$

$$\left\{\begin{aligned}&\overline{p}_{k,l,m,n}(0,s) = \eta_k\int_0^\infty \overline{p}_{0,l,m,n}(x,s)dx, \\ &k=1,2,\ldots,K,\ l=1,2,\ldots,L,\ m=0,1,\ldots,M-1,\ n=0,1,\ldots,N-1.\end{aligned}\right. \tag{4.61}$$

4242 Derivation of the solutions of the state equations with discrete transforms

The equations (4.45) - (4.61) give the Laplace transforms of the state equations, of the differential-difference equations (4.26) - (4.34) and of the boundary conditions (4.35) - (4.42), of the system. Thus the solving of the model proceeds for the present in the Laplace transform domain. From equations (4.49) - (4.53) we get by integration

$$(4.62)\qquad \bar{P}_{0,0,M,n}(x,s) = \bar{P}_{0,0,M,n}(0,s)\, e^{-sx-\int_0^x \gamma(x)dx},\ n=0,\dots,N-1,$$

$$(4.63)\qquad \begin{cases} \bar{P}_{0,l,m,N}(x,s) = \bar{P}_{0,l,m,N}(0,s)e^{-sx-\int_0^x \varphi(x)dx}, \\ l=1,2,\dots,L,\ m=0,1,\dots,M-1, \end{cases}$$

$$(4.64)\qquad \begin{cases} \bar{P}_{0,l,M,n}(x,s) = \bar{P}_{0,l,M,n}(0,s)e^{-sx-\int_0^x \gamma(x)dx}, \\ l=1,2,\dots,L,\ n=0,1,\dots,N-1, \end{cases}$$

$$(4.65)\qquad \begin{cases} \bar{P}_{k,0,m,n}(x,s) = \bar{P}_{k,0,m,n}(0,s)e^{-sx-\int_0^x \alpha_k(x)dx}, \\ k=1,2,\dots,K,\ m=0,1,\dots,M-1,\ n=0,1,\dots,N-1, \end{cases}$$

$$(4.66)\qquad \begin{cases} \bar{P}_{k,l,m,n}(x,s) = \bar{P}_{k,l,m,n}(0,s)e^{-sx-\int_0^x \alpha_k(x)dx}, \\ k=1,2,\dots,K,\ l=1,2,\dots,L,\ m=0,1,\dots,M-1,\ n=0,1,\dots,N-1. \end{cases}$$

In order to solve the equations (4.45) - (4.47), we introduce the following discrete transforms

$$(4.67)\qquad \bar{A}_i(s) = \sum_{m=0}^{M-i}\binom{M-m}{i}\bar{P}_{0,0,m,0}(s),\quad i=1,2,\dots,M,$$

$$(4.68)\qquad \begin{cases} \bar{B}_{0,i,j}(x,s) = \sum_{m=0}^{M-i}\binom{M-m}{i}\sum_{n=1}^{N-j}\binom{N-n}{j}\bar{P}_{0,0,m,n}(x,s), \\ i=1,2,\dots,M,\ j=1,2,\dots,N-1, \end{cases}$$

$$(4.69)\qquad \begin{cases} \bar{B}_{l,i,j}(x,s) = \sum_{m=0}^{M-i}\binom{M-m}{i}\sum_{n=0}^{N-j}\binom{N-n}{j}\bar{P}_{0,l,m,n}(x,s), \\ l=1,2,\dots,L,\ i=1,2,\dots,M,\ j=1,2,\dots,N. \end{cases}$$

By applying the discrete transforms (4.67) to (4.45) we obtain, after some manipulation

$$(4.70)\qquad \begin{cases} [s+\eta+\lambda+i\mu+N\nu]\bar{A}_i(s) \\ = \binom{M}{i}[1+{}_0\int^\infty \bar{P}_{0,0,M,0}(x,s)\gamma(x)dx] \\ + \sum_{m=0}^{M-i}\binom{M-m}{i}\Big[\sum_{k=1}^{K}{}_0\int^\infty \bar{P}_{k,0,m,0}(x,s)\alpha_k(x)dx \\ + \sum_{l=1}^{L}{}_0\int^\infty \bar{P}_{0,l,m,0}(x,s)\beta_l(x)dx + {}_0\int^\infty \bar{P}_{0,0,m,1}(x,s)\varphi(x)dx\Big], \\ i=1,2,\dots,M. \end{cases}$$

In (4.70) we further have, using (4.62), (4.57), (4.3) and (4.8)

$$(4.71)\qquad \begin{cases} {}_0\int^\infty \bar{P}_{0,0,M,0}(x,s)\gamma(x)dx \\ = \bar{P}_{0,0,M,0}(0,s)\,{}_0\int^\infty e^{-sx}\gamma(x)e^{-\int_0^x\gamma(x)dx}dx \\ = \mu\bar{P}_{0,0,M-1,0}(s)\,{}_0\int^\infty e^{-sx}c(x)dx = \mu\bar{c}(s)\bar{P}_{0,0,M-1,0}(s), \end{cases}$$

and using (4.65), (4.60), (4.1) and (4.8)

$$(4.72)\qquad \begin{cases} {}_0\int^\infty \bar{P}_{k,0,m,0}(x,s)\alpha_k(x)dx \\ = \bar{P}_{k,0,m,0}(0,s)\,{}_0\int^\infty e^{-sx}\alpha_k(x)e^{-\int_0^x\alpha_k(x)dx}dx \\ = \eta_k\bar{a}_k(s)\bar{P}_{0,0,m,0}(s),\ \text{whereupon} \end{cases}$$

$$(4.73)\qquad \begin{cases} \sum_{m=0}^{M-i}\binom{M-m}{i}\sum_{k=1}^{K}{}_0\int^\infty \bar{P}_{k,0,m,0}(x,s)\alpha_k(x)dx \\ = \sum_{k=1}^{K}\eta_k\bar{a}_k(s)\sum_{m=0}^{M-i}\binom{M-m}{i}\bar{P}_{0,0,m,0}(s) = \bar{A}_i(s)\sum_{k=1}^{K}\eta_k\bar{a}_k(s). \end{cases}$$

Further, using (4.69) we obtain

$$(4.74)\qquad \begin{cases} \sum_{m=0}^{M-i}\binom{M-m}{i}\sum_{l=1}^{L}{}_0\int^\infty \bar{P}_{0,l,m,0}(x,s)\beta_l(x)dx \\ = \sum_{l=1}^{L}\int_0^\infty \sum_{m=0}^{M-i}\binom{M-m}{i}\bar{P}_{0,l,m,0}(x,s)\beta_l(x)dx \\ = \sum_{l=1}^{L}\int_0^\infty \bar{B}_{l,i,N}(x,s)\beta_l(x)dx \end{cases}$$

and using (4.68) we get

$$(4.75)\quad \begin{cases} \displaystyle\sum_{m=0}^{M-i}\binom{M-m}{i}\,{}_0\!\int^{\infty}\overline{P}_{0,0,m,1}(x,s)\varphi(x)dx \\ \displaystyle= {}_0\!\int^{\infty}\sum_{m=0}^{M-i}\binom{M-m}{i}\overline{P}_{0,0,m,1}(x,s)\varphi(x)dx \\ \displaystyle= {}_0\!\int^{\infty}\overline{B}_{0,i,N-1}(x,s)\varphi(x)dx. \end{cases}$$

Thus, equation (4.70) after discrete transforms finally becomes

$$(4.76)\quad \begin{cases} \displaystyle\Big\{s + \sum_{k=1}^{K}\eta_k[1 - \overline{a}_k(s)] + \lambda + i\mu + N\nu\Big\}\overline{A}_i(s) \\ \displaystyle= \binom{M}{i}[1 + \mu\,\overline{c}(s)\overline{P}_{0,0,M-1,0}(s)] + \sum_{l=1}^{L}{}_0\!\int^{\infty}\overline{B}_{l,i,N}(x,s)\beta_l(x)dx \\ \displaystyle+ {}_0\!\int^{\infty}\overline{B}_{0,i,N-1}(x,s)\varphi(x)dx, \quad i=1,2,\ldots,M. \end{cases}$$

Applying the discrete transforms (4.68) to (4.46) we first get

$$(4.77)\quad \begin{cases} [\partial/\partial x + s + \eta + \lambda + \varphi(x)]\overline{B}_{0,i,j}(x,s) \\ \displaystyle+ \mu\sum_{m=0}^{M-i}(M-m)\binom{M-m}{i}\sum_{n=1}^{N-j}\binom{N-n}{j}\overline{P}_{0,0,m,n}(x,s) \\ \displaystyle+ \nu\sum_{m=0}^{M-i}\binom{M-m}{i}\sum_{n=1}^{N-j}(N-n)\binom{N-n}{j}\overline{P}_{0,0,m,n}(x,s) \\ \displaystyle= \mu\sum_{m=1}^{M-i}(M-m+1)\binom{M-m}{i}\sum_{n=1}^{N-j}\binom{N-n}{j}\overline{P}_{0,0,m-1,n}(x,s) \\ \displaystyle+ \nu\sum_{m=0}^{M-i}\binom{M-m}{i}\sum_{n=2}^{N-j}(N-n+1)\binom{N-n}{j}\overline{P}_{0,0,m,n-1}(x,s), \\ i=1,2,\ldots,M,\ j=1,2,\ldots,N-1, \end{cases}$$

which after some labour becomes

$$(4.78)\quad \begin{cases} [\partial/\partial x + s + \eta + \lambda + i\mu + j\nu + \varphi(x)]\overline{B}_{0,i,j}(x,s) = 0, \\ i=1,2,\ldots,M,\ j=1,2,\ldots,N-1. \end{cases}$$

By integration, equation (4.78) becomes

$$(4.79)\quad \overline{B}_{0,i,j}(x,s) = \overline{B}_{0,i,j}(0,s)e^{-(s+\eta+\lambda+i\mu+j\nu)x - \int_0^x\varphi(x)dx}, \qquad i=1,2,\ldots,M,\ j=1,2,\ldots,N-1,$$

where the $\overline{B}_{0,i,j}(0,s)$'s are as yet unknown functions of s. Applying the discrete transforms (4.69) to (4.47) we get in the same way as above

$$(4.80)\quad \begin{cases} \overline{B}_{l,i,j}(x,s) = \overline{B}_{l,i,j}(0,s)e^{-(s+\eta+i\mu+j\nu)x - \int_0^x\beta_l(x)dx}, \\ l=1,2,\ldots,L,\ i=1,2,\ldots,M,\ j=1,2,\ldots,N, \end{cases}$$

where again the functions $\overline{B}_{l,i,j}(0,s)$ are as yet unknown. As a consequence of the use of the discrete transforms (4.67) - (4.69) we have (L+1)MN unknown functions of s, viz.,

$$(4.81)\quad \begin{cases} \overline{A}_i(s), \qquad i=1,2,\ldots,M, \\ \overline{B}_{0,i,j}(0,s),\ i=1,2,\ldots,M,\ j=1,2,\ldots,N-1, \\ \overline{B}_{l,i,j}(0,s),\ i=1,2,\ldots,M,\ j=1,2,\ldots,N,\ l=1,2,\ldots,L. \end{cases}$$

The next step in deriving the solution is to try to determine the unknown functions (4.81). From equation (4.76), by substituting the expressions (4.79) and (4.80) for $\overline{B}_{0,i,j}(x,s)$ and $\overline{B}_{l,i,j}(x,s)$ respectively and using the inverse discrete transform of (4.67) for $\overline{P}_{0,0,M-1,0}(s)$, we first get, cf. equations (4.11) and (4.12)

$$(4.82)\quad \begin{cases} \displaystyle\Big\{s + \sum_{k=1}^{K}\eta_k[1 - \overline{a}_k(s)] + \lambda + i\mu + N\nu\Big\}\overline{A}_i(s) \\ \displaystyle= \binom{M}{i}[1 + \mu\,\overline{c}(s)\sum_{p=0}^{M-1}(-1)^p\binom{p+1}{p}\overline{A}_{p+1}(s)] \\ \displaystyle+ \sum_{l=1}^{L}\overline{B}_{l,i,N}(0,s)\,{}_0\!\int^{\infty}e^{-(s+\eta+i\mu+N\nu)x}\beta_l(x)e^{-\int_0^x\beta_l(x)dx}dx \\ \displaystyle+ \overline{B}_{0,i,N-1}(0,s)\,{}_0\!\int^{\infty}e^{-[s+\eta+\lambda+i\mu+(N-1)\nu]x}\varphi(x)e^{-\int_0^x\varphi(x)dx}dx, \\ i=1,2,\ldots,M, \end{cases}$$

which, after using the notation indicated, becomes

$$(4.83)\quad \begin{cases} \left\{s + \sum\limits_{k=1}^{K} \eta_k[1 - \bar{a}_k(s)] + \lambda + i\mu + N\nu\right\}\bar{A}_i(s) \\ - \binom{M}{i}[1 + \mu\,\bar{c}(s) \sum\limits_{p=1}^{M} (-1)^{p-1} p\bar{A}_p(s)] \\ = \sum\limits_{l=1}^{L} \bar{b}_l(s+\eta+i\mu+N\nu)\bar{B}_{l,i,N}(0,s) \\ + \bar{f}(s+\eta+\lambda+i\mu+[N-1]\nu)\bar{B}_{0,i,N-1}(0,s), \qquad i=1,2,\dots,M. \end{cases}$$

In (4.83) there exist, besides the functions (4.81), only known quantities of the system. In (4.83) we thus have M equations for determining the functions (4.81). From (4.54), by applying the inverse discrete transforms of (4.67) - (4.69), cf. equations (4.11) - (4.14), and using (4.65) and (4.62) and integrating, we get

$$(4.84)\quad \begin{cases} \sum\limits_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m} \sum\limits_{j=N-n}^{N-1} (-1)^{j-N+n}\binom{j}{N-n}\bar{B}_{0,i,j}(0,s) \\ = \sum\limits_{k=1}^{K} \bar{p}_{k,0,m,n}(0,s)\,\bar{a}_k(s) + \sum\limits_{l=1}^{L} \sum\limits_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m} \times \\ \sum\limits_{j=N-n}^{N} (-1)^{j-N+n}\binom{j}{N-n}\bar{B}_{l,i,j}(0,s)\bar{b}_l(s+\eta+i\mu+j\nu) \\ + \delta_{m,0}\bar{p}_{0,0,M,n}(0,s)\,\bar{c}(s) + \delta_{n,1}N\nu \sum\limits_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m}\bar{A}_i(s) \\ + (1-\delta_{n,N-1}) \sum\limits_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m} \sum\limits_{j=N-n-1}^{N-1} (-1)^{j-N+n+1} \times \\ \binom{j}{N-n-1}\bar{B}_{0,i,j}(0,s)\bar{f}(s+\eta+\lambda+i\mu+j\nu) \\ + \delta_{n,N-1}\ {}_0\!\int^{\infty} \bar{p}_{0,0,m,N}(x,s)\varphi(x)dx, \\ m=0,1,\dots,M-1, \quad n=1,2,\dots,N-1. \end{cases}$$

In (4.84), on the basis of equation (4.60) and the inverse discrete transform of $\bar{p}_{0,0,m,n}(x,s)$, we further have

$$(4.85)\quad \begin{cases} \bar{p}_{k,0,m,n}(0,s) = \eta_k\ {}_0\!\int^{\infty} \bar{p}_{0,0,m,n}(x,s)dx \\ = \eta_k \sum\limits_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m} \sum\limits_{j=N-n}^{N-1} (-1)^{j-N+n}\binom{j}{N-n}\bar{B}_{0,i,j}(0,s) \times \\ \dfrac{1 - \bar{f}(s+\eta+\lambda+i\mu+j\nu)}{s + \eta + \lambda + i\mu + j\nu}, \end{cases}$$

and on the basis of equation (4.57) and the inverse discrete transform of $\bar{p}_{0,0,M-1,n}(x,s)$ we have

$$(4.86)\quad \begin{cases} \bar{p}_{0,0,M,n}(0,s) = \mu\ {}_0\!\int^{\infty} \bar{p}_{0,0,M-1,n}(x,s)dx \\ = \mu \sum\limits_{i=1}^{M} (-1)^{i-1} i \sum\limits_{j=N-n}^{N-1} (-1)^{j-N+n}\binom{j}{N-n}\bar{B}_{0,i,j}(0,s) \times \\ \dfrac{1 - \bar{f}(s+\eta+\lambda+i\mu+j\nu)}{s + \eta + \lambda + i\mu + j\nu}. \end{cases}$$

From (4.48), by integration and using the boundary condition (4.56), we get for $\bar{p}_{0,0,m,N}(x,s)$

$$(4.87)\quad \begin{cases} \bar{p}_{0,0,m,N}(x,s) = \sum\limits_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m} \sum\limits_{j=1}^{N-1} (-1)^{j-1} \times \\ \dfrac{j\nu\bar{B}_{0,i,j}(0,s)}{\eta + \lambda + i\mu + j\nu}\left[e^{-sx - \int_0^x \varphi(x)dx} - e^{-(s+\eta+\lambda+i\mu+j\nu)x - \int_0^x \varphi(x)dx}\right], \end{cases}$$

which further gives

$$(4.88)\quad \begin{cases} {}_0\!\int^{\infty} \bar{p}_{0,0,m,N}(x,s)\varphi(x)dx = \sum\limits_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m} \sum\limits_{j=1}^{N-1} (-1)^{j-1} \times \\ \dfrac{j\nu}{\eta + \lambda + i\mu + j\nu}\,\bar{B}_{0,i,j}(0,s)\left[\bar{f}(s) - \bar{f}(s+\eta+\lambda+i\mu+j\nu)\right]. \end{cases}$$

Substituting all the results (4.85) - (4.88) in (4.84) we finally obtain

$$
(4.89)\quad \left\{
\begin{aligned}
&\sum_{i=M-m}^{M}(-1)^{i-M+m}\binom{i}{M-m}\sum_{j=N-n}^{N-1}(-1)^{j-N+n}\binom{j}{N-n}\times\\
&\left[1-\sum_{k=1}^{K}\eta_k\bar{a}_k(s)\frac{1-\bar{f}(s+\eta+\lambda+i\mu+j\nu)}{s+\eta+\lambda+i\mu+j\nu}\right]\bar{B}_{0,i,j}(0,s)\\
&+\delta_{m,0}\mu\bar{c}(s)\sum_{i=1}^{M}(-1)^{i}i\sum_{j=N-n}^{N-1}(-1)^{j-N+n}\binom{j}{N-n}\times\\
&\frac{1-\bar{f}(s+\eta+\lambda+i\mu+j\nu)}{s+\eta+\lambda+i\mu+j\nu}\bar{B}_{0,i,j}(0,s)\\
&+(1-\delta_{n,N-1})\sum_{i=M-m}^{M}(-1)^{i-M+m}\binom{i}{M-m}\times\\
&\sum_{j=N-n-1}^{N-1}(-1)^{j-N+n}\binom{j}{N-n-1}\bar{f}(s+\eta+\lambda+i\mu+j\nu)\bar{B}_{0,i,j}(0,s)\\
&+\delta_{n,N-1}\sum_{i=M-m}^{M}(-1)^{i-M+m}\binom{i}{M-m}\sum_{j=1}^{N-1}(-1)^{j}\frac{j\nu}{\eta+\lambda+i\mu+j\nu}\times\\
&\left[\bar{f}(s)-\bar{f}(s+\eta+\lambda+i\mu+j\nu)\right]\bar{B}_{0,i,j}(0,s)\\
&=\sum_{i=M-m}^{M}(-1)^{i-M+m}\binom{i}{M-m}\sum_{j=N-n}^{N}(-1)^{j-N+n}\binom{j}{N-n}\times\\
&\sum_{l=1}^{L}\bar{b}_l(s+\eta+i\mu+j\nu)\bar{B}_{l,i,j}(0,s)\\
&+\delta_{n,1}N\nu\sum_{i=M-m}^{M}(-1)^{i-M+m}\binom{i}{M-m}\bar{A}_i(s),\quad m=0,1,\dots,M-1,\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n=1,2,\dots,N-1.
\end{aligned}\right.
$$

Equation (4.89) contains only the unknown functions (4.81) and known quantities of the system. In (4.89) we thus have a further $M(N-1)$ equations for determining the unknown functions (4.81)

From 4.55), by applying (4.58), (4.59), (4.61), (4.63), (4.64), (4.66), (4.80) and the inverse discrete transforms of (4.67) - (4.69), and using similar arguments to those used for deriving equation (4.89), we obtain

$$
(4.90)\quad \left\{
\begin{aligned}
&\sum_{i=M-m}^{M}(-1)^{i-M+m}\binom{i}{M-m}\sum_{j=N-n}^{N}(-1)^{j-N+n}\binom{j}{N-n}\times\\
&\left[1-\sum_{k=1}^{K}\eta_k\bar{a}_k(s)\frac{1-\bar{b}_l(s+\eta+i\mu+j\nu)}{s+\eta+i\mu+j\nu}\right]\bar{B}_{l,i,j}(0,s)\\
&+\delta_{m,0}\mu\bar{c}(s)\sum_{i=1}^{M}(-1)^{i}i\sum_{j=N-n}^{N}(-1)^{j-N+n}\binom{j}{N-n}\times\\
&\frac{1-\bar{b}_l(s+\eta+i\mu+j\nu)}{s+\eta+i\mu+j\nu}\bar{B}_{l,i,j}(0,s)\\
&+\delta_{n,N-1}\nu\bar{f}(s)\sum_{i=M-m}^{M}(-1)^{i-M+m}\binom{i}{M-m}\sum_{j=1}^{N}(-1)^{j}j\times\\
&\frac{1-\bar{b}_l(s+\eta+i\mu+j\nu)}{s+\eta+i\mu+j\nu}\bar{B}_{l,i,j}(0,s)\\
&=(1-\delta_{n,0})\lambda_l\sum_{i=M-m}^{M}(-1)^{i-M+m}\binom{i}{M-m}\sum_{j=N-n}^{N-1}(-1)^{j-N+n}\binom{j}{N-n}\times\\
&\frac{1-\bar{f}(s+\eta+\lambda+i\mu+j\nu)}{s+\eta+\lambda+i\mu+j\nu}\bar{B}_{0,i,j}(0,s)\\
&+\delta_{n,0}\lambda_l\sum_{i=M-m}^{M}(-1)^{i-M+m}\binom{i}{M-m}\bar{A}_i(s),\\
&l=1,2,\dots,L,\ m=0,1,\dots,M-1,\ n=0,1,\dots,N-1.
\end{aligned}\right.
$$

In (4.90) we have LMN independent equations for the unknown functions in (4.81). For the (L+1)MN unknown functions in (4.81) we thus have altogether (L+1)MN independent equations: M equations in (4.83), M(N-1) equations in (4.89) and LMN equations in (4.90). All these equations are linear algebraic equations of the unknown functions (4.81). We can thus consider the functions solved in principle.[1] Due to the complicated expressions, in the general case the solution of the equation system will not, however, be derived explicitly.

1 The solutions exist at least for some (if, perhaps, not for all) values of s, Re(s) > 0. It can be shown, for example, that with exponential distributions for all repair times of the system, the solutions exist for all s, Re(s) > 0.

4243 Transient state solutions of the state equations in the Laplace transform domain

After the functions (4.81) have been derived from the equations (4.83), (4.89) and (4.90), the Laplace transforms of the transient state solutions of the state probabilities may easily be obtained. For $\bar{P}_{0,0,m,0}(s)$, $m=0,1,\dots,M-1$, we get directly, as the inverse discrete transforms of $\bar{A}_i(s)$, $i=1,2,\dots,M$,

$$\bar{P}_{0,0,m,0}(s) = \sum_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m}\bar{A}_i(s), \quad m=0,1,\dots,M-1. \tag{4.91}$$

For $\bar{P}_{0,0,m,n}(s)$, $m=0,1,\dots,M-1$, $n=1,2,\dots,N-1$, by applying the Laplace transform of equation (4.7), and expressing $\bar{P}_{0,0,m,n}(x,s)$ as inverse discrete transforms of $\bar{B}_{0,i,j}(x,s)$, $i=1,2,\dots,M$, $j=1,2,\dots,N-1$, using (4.79) and integrating, we get

$$\begin{cases} \bar{P}_{0,0,m,n}(s) = \displaystyle\sum_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m}\sum_{j=N-n}^{N-1}(-1)^{j-N+n}\binom{j}{N-n}\times \\ \dfrac{1-\bar{f}(s+\eta+\lambda+i\mu+j\nu)}{s+\eta+\lambda+i\mu+j\nu}\bar{B}_{0,i,j}(0,s), \quad m=0,1,\dots,M-1, \\ \qquad n=1,2,\dots,N-1. \end{cases} \tag{4.92}$$

By using a relation analogous to equation (4.7), substituting the expression (4.87) and integrating, $\bar{P}_{0,0,m,N}(s)$ becomes

$$\begin{cases} \bar{P}_{0,0,m,N}(s) = \displaystyle\sum_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m}\sum_{j=1}^{N-1}(-1)^{j-1}\times \\ \left\{\dfrac{j\nu[1-\bar{f}(s)]}{s(s+\eta+\lambda+i\mu+j\nu)} - \dfrac{j\nu[\bar{f}(s)-\bar{f}(s+\eta+\lambda+i\mu+j\nu)]}{(\eta+\lambda+i\mu+j\nu)(s+\eta+\lambda+i\mu+j\nu)}\right\}\times \\ \bar{B}_{0,i,j}(0,s), \quad m=0,1,\dots,M-1. \end{cases} \tag{4.93}$$

From (4.62) and (4.57) we get for $\bar{P}_{0,0,M,0}(s)$, as above

$$\bar{P}_{0,0,M,0}(s) = \mu\frac{1-\bar{c}(s)}{s}\sum_{i=1}^{M}(-1)^{i-1}\, i\,\bar{A}_i(s). \tag{4.94}$$

By similar argumentation we can obtain for the remainder of the state probabilities the expressions of their Laplace transforms. These are

$$\begin{cases} \bar{P}_{0,0,M,n}(s) = \mu\dfrac{1-\bar{c}(s)}{s}\displaystyle\sum_{i=1}^{M}(-1)^{i-1}\, i\sum_{j=N-n}^{N-1}(-1)^{j-N+n}\times \\ \dbinom{j}{N-n}\dfrac{1-\bar{f}(s+\eta+\lambda+i\mu+j\nu)}{s+\eta+\lambda+i\mu+j\nu}\bar{B}_{0,i,j}(0,s), \quad n=1,2,\dots,N-1, \end{cases} \tag{4.95}$$

$$\begin{cases} \bar{P}_{0,l,m,n}(s) = \displaystyle\sum_{i=M-m}^{M}(-1)^{i-M+m}\binom{i}{M-m}\sum_{j=N-n}^{N}(-1)^{j-N+n}\binom{j}{N-n}\times \\ \dfrac{1-\bar{b}_l(s+\eta+i\mu+j\nu)}{s+\eta+i\mu+j\nu}\bar{B}_{l,i,j}(0,s), \quad l=1,2,\dots,L,\ m=0,1,\dots,M-1, \\ \qquad n=0,1,\dots,N-1, \end{cases} \tag{4.96}$$

$$\begin{cases} \bar{P}_{0,l,m,N}(s) = \nu\dfrac{1-\bar{f}(s)}{s}\displaystyle\sum_{i=M-m}^{M}(-1)^{i-M+m}\binom{i}{M-m}\sum_{j=1}^{N}(-1)^{j-1}\times \\ j\,\dfrac{1-\bar{b}_l(s+\eta+i\mu+j\nu)}{s+\eta+i\mu+j\nu}\bar{B}_{l,i,j}(0,s), \quad l=1,2,\dots,L,\ m=0,1,\dots,M-1, \end{cases} \tag{4.97}$$

$$\begin{cases} \bar{P}_{0,l,M,n}(s) = \mu\dfrac{1-\bar{c}(s)}{s}\displaystyle\sum_{i=1}^{M}(-1)^{i-1}\, i\sum_{j=N-n}^{N}(-1)^{j-N+n}\times \\ \dbinom{j}{N-n}\dfrac{1-\bar{b}_l(s+\eta+i\mu+j\nu)}{s+\eta+i\mu+j\nu}\bar{B}_{l,i,j}(0,s), \quad l=1,2,\dots,L, \\ \qquad n=0,1,\dots,N-1, \end{cases} \tag{4.98}$$

$$\begin{cases} \bar{P}_{k,0,m,0}(s) = \eta_k\dfrac{1-\bar{a}_k(s)}{s}\displaystyle\sum_{i=M-m}^{M}(-1)^{i-M+m}\binom{i}{M-m}\bar{A}_i(s), \\ k=1,2,\dots,K,\ m=0,1,\dots,M-1, \end{cases} \tag{4.99}$$

$$\begin{cases} \bar{P}_{k,0,m,n}(s) = \eta_k\dfrac{1-\bar{a}_k(s)}{s}\displaystyle\sum_{i=M-m}^{M}(-1)^{i-M+m}\binom{i}{M-m}\times \\ \displaystyle\sum_{j=N-n}^{N-1}(-1)^{j-N+n}\binom{j}{N-n}\dfrac{1-\bar{f}(s+\eta+\lambda+i\mu+j\nu)}{s+\eta+\lambda+i\mu+j\nu}\bar{B}_{0,i,j}(0,s), \\ k=1,2,\dots,K,\ m=0,1,\dots,M-1,\ n=1,2,\dots,N-1, \end{cases} \tag{4.100}$$

$$\begin{cases} \bar{P}_{k,l,m,n}(s) = \eta_k\dfrac{1-\bar{a}_k(s)}{s}\displaystyle\sum_{i=M-m}^{M}(-1)^{i-M+m}\binom{i}{M-m}\times \\ \displaystyle\sum_{j=N-n}^{N}(-1)^{j-N+n}\binom{j}{N-n}\dfrac{1-\bar{b}_l(s+\eta+i\mu+j\nu)}{s+\eta+i\mu+j\nu}\bar{B}_{l,i,j}(0,s), \\ k=1,2,\dots,K,\ l=1,2,\dots,L,\ m=0,1,\dots,M-1,\ n=0,1,\dots,N-1. \end{cases} \tag{4.101}$$

Equations (4.91) - (4.101) now give the Laplace transforms of the transient state probabilities of the system. In the general case, i.e. if the expressions for the repair rates $\alpha_k(x)$, k=1, 2,...K, $\beta_l(x)$, l=1,2,...,L, $\gamma(x)$ and $\varphi(x)$ are not known, the expressions for the state probabilities naturally cannot be obtained. But with given values of the repair rates, equations (4.91) - (4.101) may be inverted to give the transient state probabilities.

4244 Steady state behaviour of the system

The asymptotic or steady state behaviour of the system, i.e. the behaviour of the system when the time that has elapsed since the system was first put into operation becomes long enough, may be derived from equations (4.91) - (4.101) by using equation (4.10), i.e. the final value theorem of Laplace transforms. It is to be noted that the steady state behaviour of the system may be obtained, also in the general case, in the form of state probabilities and not only as Laplace transforms of these probabilities. In the derivation of the steady state probabilities the result of the following lemma will also be used.[1]

Lemma 4.1. Let a(x) be the density function of a positive random variable X, such that

(i) $E(X) = \int_0^\infty x\, a(x)\, dx$ exists, and

(ii) $\lim_{x\to\infty} \left\{x[1 - A(x)]\right\} = \lim_{x\to\infty} \left\{x\left[1 - \int_0^x a(x)dx\right]\right\} = 0.$

Then we have, when $\bar{a}(s)$ is the Laplace transform of a(x),

$$\lim_{s\to 0} \frac{1 - \bar{a}(s)}{s} = E(X). \tag{4.102}$$

Applying the equations (4.10) and (4.102) we get for the steady state probabilities of the system, the following expressions

1 For the proof of the lemma see VIRTANEN (1974b), p. 48.

(if the limiting values appearing in these expressions exist).

$$(4.103)\begin{cases}\tilde{P}_{0,0,m,0} = \lim_{t\to\infty} P_{0,0,m,0}(t) = \lim_{s\to 0} s\,\bar{P}_{0,0,m,0}(s) \\ = \sum_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m}\tilde{A}_i, \quad m=0,1,\dots,M-1,\end{cases}$$

$$(4.104)\begin{cases}\tilde{P}_{0,0,m,n} = \lim_{t\to\infty} P_{0,0,m,n}(t) = \lim_{s\to 0} s\,\bar{P}_{0,0,m,n}(s) \\ = \sum_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m}\sum_{j=N-n}^{N-1} (-1)^{j-N+n}\binom{j}{N-n} \times \\ \frac{1-\bar{f}(\eta+\lambda+i\mu+j\nu)}{\eta+\lambda+i\mu+j\nu}\tilde{B}_{0,i,j}, \quad m=0,1,\dots,M-1,\ n=1,2,\dots,N-1,\end{cases}$$

$$(4.105)\begin{cases}\tilde{P}_{0,0,m,N} = \lim_{t\to\infty} P_{0,0,m,N}(t) = \lim_{s\to 0} s\,\bar{P}_{0,0,m,N}(s) \\ = \sum_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m}\sum_{j=1}^{N-1} (-1)^{j-1}\frac{j\nu}{\eta+\lambda+i\mu+j\nu} \times \\ \left\{E_f - \frac{1-\bar{f}(\eta+\lambda+i\mu+j\nu)}{\eta+\lambda+i\mu+j\nu}\right\}\tilde{B}_{0,i,j}, \quad m=0,1,\dots,M-1,\end{cases}$$

$$(4.106)\begin{cases}\tilde{P}_{0,0,M,0} = \lim_{t\to\infty} P_{0,0,M,0}(t) = \lim_{s\to 0} s\,\bar{P}_{0,0,M,0}(s) \\ = \mu E_c \sum_{i=1}^{M} (-1)^{i-1} i\,\tilde{A}_i,\end{cases}$$

$$(4.107)\begin{cases}\tilde{P}_{0,0,M,n} = \lim_{t\to\infty} P_{0,0,M,n}(t) = \lim_{s\to 0} s\,\bar{P}_{0,0,M,n}(s) \\ = \mu E_c \sum_{i=1}^{M} (-1)^{i-1} i \sum_{j=N-n}^{N-1} (-1)^{j-N+n}\binom{j}{N-n} \times \\ \frac{1-\bar{f}(\eta+\lambda+i\mu+j\nu)}{\eta+\lambda+i\mu+j\nu}\tilde{B}_{0,i,j}, \quad n=0,1,\dots,N-1,\end{cases}$$

$$(4.108)\quad \begin{cases} \tilde{P}_{0,l,m,n} = \lim\limits_{t\to\infty} P_{0,l,m,n}(t) = \lim\limits_{s\to 0} s\,\bar{P}_{0,l,m,n}(s) \\ = \sum\limits_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m} \sum\limits_{j=N-n}^{N} (-1)^{j-N+n}\binom{j}{N-n} \times \\ \dfrac{1-\bar{b}_l(\eta+i\mu+j\nu)}{\eta+i\mu+j\nu}\,\tilde{B}_{l,i,j}, \\ l=1,2,\ldots,L,\ m=0,1,\ldots,M-1,\ n=0,1,\ldots,N-1, \end{cases}$$

$$(4.109)\quad \begin{cases} \tilde{P}_{0,l,m,N} = \lim\limits_{t\to\infty} P_{0,l,m,N}(t) = \lim\limits_{s\to 0} s\,\bar{P}_{0,l,m,N}(s) \\ = \nu E_f \sum\limits_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m} \sum\limits_{j=1}^{N} (-1)^{j-1}\, j\, \dfrac{1-\bar{b}_l(\eta+i\mu+j\nu)}{\eta+i\mu+j\nu} \times \\ \tilde{B}_{l,i,j}, \quad l=1,2,\ldots,L,\ m=0,1,\ldots,M-1, \end{cases}$$

$$(4.110)\quad \begin{cases} \tilde{P}_{0,l,M,n} = \lim\limits_{t\to\infty} P_{0,l,M,n}(t) = \lim\limits_{s\to 0} s\,\bar{P}_{0,l,M,n}(s) \\ = \mu E_c \sum\limits_{i=1}^{M} (-1)^{i-1}\, i \sum\limits_{j=N-n}^{N} (-1)^{j-N+n}\binom{j}{N-n} \dfrac{1-\bar{b}_l(\eta+i\mu+j\nu)}{\eta+i\mu+j\nu} \times \\ \tilde{B}_{l,i,j}, \quad l=1,2,\ldots,L,\ n=0,1,\ldots,N-1, \end{cases}$$

$$(4.111)\quad \begin{cases} \tilde{P}_{k,0,m,0} = \lim\limits_{t\to\infty} P_{k,0,m,0}(t) = \lim\limits_{s\to 0} s\,\bar{P}_{k,0,m,0}(s) \\ = \eta_k E_{a_k} \sum\limits_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m}\tilde{A}_i, \quad m=0,1,\ldots,M-1, \end{cases}$$

$$(4.112)\quad \begin{cases} \tilde{P}_{k,0,m,n} = \lim\limits_{t\to\infty} P_{k,0,m,n}(t) = \lim\limits_{s\to 0} s\,\bar{P}_{k,0,m,n}(s) \\ = \eta_k E_{a_k} \sum\limits_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m} \sum\limits_{j=N-n}^{N-1} (-1)^{j-N+n}\binom{j}{N-n} \times \\ \dfrac{1-\bar{f}(\eta+\lambda+i\mu+j\nu)}{\eta+\lambda+i\mu+j\nu}\,\tilde{B}_{0,i,j}, \\ k=1,2,\ldots,K,\ m=0,1,\ldots,M-1,\ n=1,2,\ldots,N-1, \end{cases}$$

$$(4.113)\quad \begin{cases} \tilde{P}_{k,l,m,n} = \lim\limits_{t\to\infty} P_{k,l,m,n}(t) = \lim\limits_{s\to 0} s\,\bar{P}_{k,l,m,n}(s) \\ = \eta_k E_{a_k} \sum\limits_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m} \sum\limits_{j=N-n}^{N} (-1)^{j-N+n}\binom{j}{N-n} \times \\ \dfrac{1-\bar{b}_l(\eta+i\mu+j\nu)}{\eta+i\mu+j\nu}\,\tilde{B}_{l,i,j}, \\ k=1,2,\ldots,K,\ l=1,2,\ldots,L,\ m=0,1,\ldots,M-1,\ n=0,1,\ldots,N-1. \end{cases}$$

In equations (4.103) - (4.113) above we have introduced the following notation not defined earlier

$$(4.114)\quad \tilde{A}_i = \lim_{s\to 0} s\,\bar{A}_i(s), \qquad i=1,2,\ldots,M,$$

$$(4.115)\quad \tilde{B}_{0,i,j} = \lim_{s\to 0} s\,\bar{B}_{0,i,j}(0,s), \qquad i=1,2,\ldots,M,\ j=1,2,\ldots,N-1,$$

$$(4.116)\quad \tilde{B}_{l,i,j} = \lim_{s\to 0} s\,\bar{B}_{l,i,j}(0,s), \qquad l=1,2,\ldots,L,\ i=1,2,\ldots,M,\ j=1,2,\ldots,N,$$

$$(4.117)\quad E_{a_k} = \int_0^\infty x\,a_k(x)\,dx, \qquad k=1,2,\ldots,K,$$

$$(4.118)\quad E_c = \int_0^\infty x\,c(x)\,dx,$$

$$(4.119)\quad E_f = \int_0^\infty x\,f(x)\,dx.$$

43 Reliability analysis of the system

431 Laplace transforms of the transient state reliability characteristics of the system

The system under study is a system which also possesses the property of operation with reduced efficiency, even at several different levels of performance. Because of this we introduce for the reliability analysis of the system the new extended concepts of reliability developed in Chapter 3. These generalized characteristics of reliability now become functions, not only of time, but also of the level of performance of the system (excluding A_C, the mean availability of the capacity, which is a weighted sum in respect of the performance level and thus a function of time only).

However, the use of the new concepts enables us to remain within the bounds of the traditional concepts of reliability. The use of these concepts only obliges us to consider the system as a two-stage operable or inoperable system, by classifying the states with reduced efficiency either as operable or as inoperable states (i.e. some of them operable and the others inoperable). Thereafter the values of the characteristics of traditional reliability are obtained as particular cases of the new, more comprehensive characteristics, i.e. as their values at that level of performance which has been defined as the minimum value for an operable state.

As we saw in Chapter 3 derivation of the reliability characteristics can be based on the possible levels of performance (or on the corresponding proportional levels) and on the probability of their occurrence. The probabilities of the different performance levels for their part can be determined with the help of the state probabilities of the system. The states of the system given in Table 4.2 were defined precisely from the point of view of the level of performance of the system.

The level of performance of the system amounts to full capacity only when the system is in state 1, when all the four subsystems

are operable, i.e. when all the components of S_1, S_2 and S_4 are operable and m ($0 \leq m \leq M-1$) of the M components in S_3 have failed. Whenever it is in any of the states 5-14 the system is totally inoperable, due to the failure of one of the subsystems (there may possibly be additional failures also among the components of the other subsystems). In states 2, 3 and 4 the system operates with reduced efficiency, i.e. the level of performance of the system is positive, but below full capacity.

Let the level of performance of the system (at time t) be denoted by C(t) and let the different reduced values of C(t) be

(4.120) $c_{0,n}$, $n=1,2,\dots,N-1$ (system in state 2),

(4.121) $c_{l,0}$, $l=1,2,\dots,L$ (system in state 3),

(4.122) $c_{l,n}$, $l=1,2,\dots,L$, $n=1,2,\dots,N-1$ (system in state 4).

Using the notation introduced in section 4231 we can further specify

(4.123) $C(t) = c_{0,n}$, if $K(t)=L(t)=0$, $0 \leq M(t) \leq M-1$, $1 \leq N(t)=n \leq N-1$,

(4.124) $C(t) = c_{l,0}$, if $K(t)=0$, $1 \leq L(t)=l \leq L, 0 \leq M(t) \leq M-1, N(t)=0$,

(4.125) $C(t) = c_{l,n}$, if $K(t)=0$, $1 \leq L(t)=l \leq L, 0 \leq M(t) \leq M-1, 1 \leq N(t)= n \leq N-1$.

The values of the reduced performance levels $c_{0,n}$, $c_{l,0}$, $c_{l,n}$, $l=1,2,\dots,L$, $n=1,2\dots,N-1$, depend on the internal structural details and the principle of operation of the system. All that can generally be said about them is that $c_{0,n}$ and $c_{l,n}$ are non-increasing with increasing n and that

(4.126) $0 < c_{0,n}, c_{l,0}, c_{l,n} < C$, $l=1,2,\dots,L$, $n=1,2,\dots,N-1$,

where C stands for the full capacity of the system. However, for $c_{0,n}$ the relation

(4.127) $c_{0,n} = \left(1 - \frac{n}{N}\right)C$, $n= 1,2,\dots,N-1$,

is the most natural. Because the components of S_2 may all be different, also the reduced performance levels $c_{l,0}$, $l=1,2,\ldots,L$ may be different. When the system is in state 4 both subsystem S_2 and subsystem S_4 are able to operate only with reduced efficiency. The level of performance of the whole system, $c_{l,n}$, is then the minimum of the levels of performance of the subsystems S_2 and S_4 or

$$c_{l,n} = \min\left\{c_{l,0},\ c_{0,n}\right\},\ l=1,2,\ldots,L,\ n=1,2,\ldots,N-1. \tag{4.128}$$

In certain cases, however, the relation

$$c_{l,n} = (c_{l,0}/C)c_{0,n},\ l=1,2,\ldots,L,\ n=1,2,\ldots,N-1 \tag{4.129}$$

may be more relevant. The final determination of $c_{l,n}$ must therefore be made one case at a time.

The proportional level of performance of the system is equal to 1 when the system is in state 1, equal to 0 when the system is in one of the states 5-14, and positive, but less than 1 when the system is in one of the states 2-4. In states 2-4 we denote the proportional level of performance of the system by

$$w_{0,n},\ n=1,2,\ldots,N-1 \quad \text{(system in state 2)}, \tag{4.130}$$

$$w_{l,0},\ l=1,2,\ldots,L \quad \text{(system in state 3)}, \tag{4.131}$$

$$w_{l,n},\ l=1,2,\ldots,L,\ n=1,2,\ldots,N-1 \quad \text{(system in state 4)}. \tag{4.132}$$

The relations (4.126) - (4.129), expressed now as relations for the proportional levels of performance, become

$$0 < w_{0,n},\ w_{l,0},\ w_{l,n} < 1,\ l=1,2,\ldots,L,\ n=1,2,\ldots,N-1, \tag{4.133}$$

$$w_{0,n} = 1 - \frac{n}{N},\ n=1,2,\ldots,N-1, \tag{4.134}$$

$$w_{l,n} = \min\left\{w_{l,0}, w_{0,n}\right\},\ l=1,2,\ldots,L,\ n=1,2,\ldots,N-1, \tag{4.135}$$

$$w_{l,n} = w_{l,0}\cdot w_{0,n},\ l=1,2,\ldots,L,\ n=1,2,\ldots,N-1. \tag{4.136}$$

We now know all the quantities required for obtaining the reliability characteristics of the system. The levels of performance in the different states of the system are assumed either to be known in advance or to be determinable from the structural and operational properties of the system. The state probabilities were obtained as a result of the analysis in paragraph 4.2. As far as the transient or time-dependent state of the system is concerned we know, in the general case, only the Laplace transforms of the state probabilities. This of course means that in the general case we can also derive expressions only for the Laplace transforms of the reliability characteristics. With given values for the unknown quantities of the system, i.e. with given failure and repair time distributions, these expressions may then be inverted to give the reliability characteristics themselves.

4311 Availability of levels of performance

The value of A_0, availability of levels of performance, at time t ($t \geq 0$) and at performance level c ($0 < c \leq C$) was defined in Chapter 3 as the probability that at time t the system is operable at a level of performance that is not less than c. The Laplace transform of $A_0(c,t)$, denoted $\bar{A}_0(c,s)$, thus becomes for the system under study

$$\bar{A}_0(c,s) = \sum_{m=0}^{M-1}\Big[\bar{P}_{0,0,m,0}(s) + \sum_{c_{0,n}\geq c}\bar{P}_{0,0,m,n}(s) + \sum_{c_{l,0}\geq c}\bar{P}_{0,l,m,0}(s) + \sum_{c_{l,n}\geq c}\bar{P}_{0,l,m,n}(s)\Big], \tag{4.137}$$

where the Laplace transforms of the state probabilities are given in equations (4.91), (4.92) and (4.96). For the first sum on the right hand side of equation (4.137), using equation (4.91) and simplifying we obtain

$$(4.138)\quad \begin{cases} \sum_{m=0}^{M-1} \overline{F}_{0,0,m,0}(s) = \sum_{m=0}^{M-1} \sum_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m}\overline{A}_i(s) \\ = \sum_{i=1}^{M} \sum_{m=M-i}^{M-1} (-1)^{i-M+m}\binom{i}{M-m}\overline{A}_i(s) \\ = \sum_{i=1}^{M} \overline{A}_i(s) \sum_{r=1}^{i} (-1)^{i-r}\binom{i}{r} \\ = \sum_{i=1}^{M} \overline{A}_i(s)\left[\sum_{r=0}^{i} (-1)^{i-r}\binom{i}{r} - (-1)^i\right] \\ = \sum_{i=1}^{M} (-1)^{i-1}\overline{A}_i(s). \end{cases}$$

Reasoning in the same manner as in (4.138) we finally get for $\overline{A}_0(c,s)$

$$(4.139)\quad \begin{cases} \overline{A}_0(c,s) = \sum_{i=1}^{M} (-1)^{i-1}\overline{A}_i(s) + \sum_{i=1}^{M} (-1)^{i-1} \sum_{c_{0,n}\geq c} \sum_{j=N-n}^{N-1} (-1)^{j-N+n} \times \\ \binom{j}{N-n}\frac{1-\overline{f}(s+\eta+\lambda+i\mu+j\nu)}{s+\eta+\lambda+i\mu+j\nu}\,\overline{B}_{0,i,j}(0,s) \\ + \sum_{i=1}^{M} (-1)^{i-1} \sum_{c_{1,0}\geq c} \frac{1-\overline{b}_1(s+\eta+i\mu+N\nu)}{s+\eta+i\mu+N\nu}\,\overline{B}_{1,i,N}(0,s) \\ + \sum_{i=1}^{M} (-1)^{i-1} \sum_{c_{1,n}\geq c} \sum_{j=N-n}^{N} (-1)^{j-N+n}\binom{j}{N-n} \times \\ \frac{1-\overline{b}_1(s+\eta+i\mu+j\nu)}{s+\eta+i\mu+j\nu}\,\overline{B}_{1,i,j}(0,s). \end{cases}$$

We thus find that $\overline{A}_0(c,s)$ may be expressed with the help of the functions $\overline{A}_i(s)$, $\overline{B}_{0,i,j}(0,s)$ and $\overline{B}_{1,i,j}(0,s)$, which were the original solutions of the model. It is therefore not necessary to determine the state probabilities (or more correctly, their Laplace transforms) at all, because $\overline{A}_0(c,s)$ may be obtained directly on the basis of the solutions of equations (4.83), (4.89) and (4.90).

As particular values of $\overline{A}_0(c,s)$, for example when $c = C$ (the capacity of the system), we have

$$(4.140)\quad \overline{A}_0(C,s) = \sum_{i=1}^{M} (-1)^{i-1}\overline{A}_i(s),$$

which gives the Laplace transform of $A_0(C,t)$, i.e. the probability that the system is operable with full capacity; and when c has the value

$$(4.141)\quad \hat{c} = \min_{\substack{l=1,2,\dots,L \\ n=1,2,\dots,N-1}} \left\{c_{0,n},\ c_{l,0},\ c_{l,n}\right\},$$

$$(4.142)\quad \begin{cases} \overline{A}_0(\hat{c},s) = \sum_{i=1}^{M} (-1)^{i-1}\overline{A}_i(s) + \sum_{i=1}^{M} (-1)^{i-1} \sum_{j=1}^{N-1} (-1)^{j-1} \times \\ \frac{1-\overline{f}(s+\eta+\lambda+i\mu+j\nu)}{s+\eta+\lambda+i\mu+j\nu}\,\overline{B}_{0,i,j}(0,s) \\ + \sum_{i=1}^{M} (-1)^{i-1} \sum_{j=1}^{N} (-1)^{j-1} \sum_{l=1}^{L} \frac{1-\overline{b}_l(s+\eta+i\mu+j\nu)}{s+\eta+i\mu+j\nu}\,\overline{B}_{l,i,j}(0,s), \end{cases}$$

which gives the Laplace transform of the probability that the system is operable, either with full capacity or with reduced efficiency. The two particular values of the generalized availability function, $A_0(C,t)$ and $A_0(\hat{c},t)$, which are the inverted values of $\overline{A}_0(C,s)$ and $\overline{A}_0(\hat{c},s)$, also have an interesting interpretation in the light of the traditional concept of availability. $A_0(C,t)$ expresses the availability of the system in the case when also the failures resulting in a reduction in the level of performance are classified as system failures; and $A_0(\hat{c},t)$ expresses the availability of the system in the case when failures resulting in a reduction in the level of performance are not regarded as failures at all.

4312 Mean availability of the capacity

Calculation of A_C, the mean availability of the capacity, is based on equation (3.23), in which this characteristic is expressed as a weighted sum of the state probabilities. For the system under study this equation becomes (now in the Laplace transform domain)

$$(4.143)\quad \begin{cases} \bar{A}_C(s) = \sum_{m=0}^{M-1} \bar{P}_{0,0,m,0}(s) + \sum_{l=1}^{L}\sum_{m=0}^{M-1} \bar{P}_{0,l,m,0}(s)\, w_{l,0} \\ + \sum_{m=0}^{M-1}\sum_{n=1}^{N-1} \bar{P}_{0,0,m,n}(s)\, w_{0,n} + \sum_{l=1}^{L}\sum_{m=0}^{M-1}\sum_{n=1}^{N-1} \bar{P}_{0,l,m,n}(s)\, w_{l,n}. \end{cases}$$

Using equations (4.91), (4.92) and (4.96) we can obtain after simplification an equation where $\bar{A}_C(s)$ is expressed in terms of $\bar{A}_i(s)$, $\bar{B}_{0,i,j}(0,s)$ and $\bar{B}_{l,i,j}(0,s)$

$$(4.144)\quad \begin{cases} \bar{A}_C(s) = \sum_{i=1}^{M} (-1)^{i-1}\bar{A}_i(s) \\ + \sum_{i=1}^{M} (-1)^{i-1} \sum_{n=1}^{N-1} w_{0,n} \sum_{j=N-n}^{N-1} (-1)^{j-N+n} \binom{j}{N-n} \times \\ \dfrac{1 - \bar{f}(s+\eta+\lambda+i\mu+j\nu)}{s+\eta+\lambda+i\mu+j\nu}\, \bar{B}_{0,i,j}(0,s) \\ + \sum_{i=1}^{M} (-1)^{i-1} \sum_{n=0}^{N-1}\sum_{l=1}^{L} w_{l,n} \sum_{j=N-n}^{N} (-1)^{j-N+n}\binom{j}{N-n} \times \\ \dfrac{1 - \bar{b}_l(s+\eta+i\mu+j\nu)}{s+\eta+i\mu+j\nu}\, \bar{B}_{l,i,j}(0,s). \end{cases}$$

Here too, if we want to remain within the bounds of the traditional concepts this is also possible by using equation (4.144). We simply make the original reduced proportional levels of performance equal to 0 or 1 according to whether the failures causing reduction in the efficiency of the system are regarded as system failures or not. As extreme values for this traditional availability, by making the quantities $w_{0,n}$, $w_{l,0}$, $w_{l,n}$, $l=1,2,\dots,L$, $n=1,2,\dots,N-1$, either all equal to 0 or all equal to 1, we get respectively

$$(4.145)\quad \bar{A}_C^{(0)}(s) = \sum_{i=1}^{M} (-1)^{i-1}\bar{A}_i(s),$$

or

$$(4.146)\quad \begin{cases} \bar{A}_C^{(1)}(s) = \sum_{i=1}^{M} (-1)^{i-1}\bar{A}_i(s) \\ + \sum_{i=1}^{M} (-1)^{i-1} \sum_{j=1}^{N-1} (-1)^{j-1} \dfrac{1 - \bar{f}(s+\eta+\lambda+i\mu+j\nu)}{s+\eta+\lambda+i\mu+j\nu}\, \bar{B}_{0,i,j}(0,s) \\ + \sum_{i=1}^{M} (-1)^{i-1} \sum_{j=1}^{N} (-1)^{j-1} \sum_{l=1}^{L} \dfrac{1-\bar{b}_l(s+\eta+i\mu+j\nu)}{s+\eta+i\mu+j\nu}\, \bar{B}_{l,i,j}(0,s). \end{cases}$$

As can be seen, the expressions in (4.145) and (4.146) coincide with those in (4.140) and (4.142). This is natural, for they give the Laplace transforms of the probabilities of the same things.

4313 Reliability of levels of performance

The value of the generalized reliability characteristic R_0, i.e. the reliability of levels of performance, at time t ($t \geq 0$) and at performance level c, ($0 < c \leq C$) was defined in Chapter 3 as the probability that the level of performance of the system never becomes less than c during the interval [0,t]. In order to obtain the value of $R_0(c,t)$ from the state probabilities, we must redefine the states of the system, as was shown in 3224. All those states in which the level of performance of the system is less than c are defined as absorbing states. All the other states are left unchanged. We then get $R_0(c,t)$ as the probability that the system at time t is in one of the non-absorbing states, i.e. as the sum of the state probabilities of the non-absorbing states. According to the property of the absorbing state, this probability is equivalent to the probability that the level of performance of the system will not fall below c during the interval [0,t].

The introduction of absorbing states leads, however, to some modifications of the preceding "availability" model. These modifications depend on the value of c, and in such a way that in the general case (when the value of c is not fixed) the model cannot be built up. But as soon as c has been fixed the model can be formulated and solved in a manner similar to that used earlier for the availability model. However, the modifications in the model, i.e. in the equations (4.26)-(4.44), have certain general principles. The modifications can be achieved by one of the following measures:

1 A state can be made an absorbing state by making the repair rate in that state equal to 0: after the system has got into a state where the level of performance is less than c, the repair of the system is no longer taken into account. The states 6, 7, 9, 10, 11, 12, 13 and 14 are made as absorbing states by setting $\alpha_k(x) \equiv 0$, $k=1,2,\ldots,K$, and $\gamma(x) \equiv 0$ everywhere in the equations (4.26)-(4.44). It will be noted that the states 6 and 7 and 9-14 are absorbing states for all values of c, because these states indicate a system failure. States 5 and 8 are made absorbing states by making $\varphi(x) \equiv 0$ in all those cases where the repair of subsystem S_4, with all the N components failed, is considered, i.e. in equations (4.29) and (4.31) and for n=N-1 also in equations (4.35) and (4.36). Thus also states 5 and 8 are always made absorbing states. Modifications of the first type are easy to carry out because the structure of the model does not change. We can use the results of the preceding availability model by making the appropriate substitutions in these results.

2 States 2-4 may be absorbing or non-absorbing, depending on the relations between the value of c and the levels of performance $c_{0,n}$, $c_{1,0}$ and $c_{1,n}$, $1=1,2,\ldots,L$, $n=1,2,\ldots,N-1$ in these states. This is revealed especially by the fact that for some values of 1 and n these states (i.e. certain substates of the superstates 2 to 4) are absorbing and that for the remaining values of 1 and n the states remain non-absorbing. Making the states in question absorbing now requires structural modifications of the model. There is now a change in the values of 1 and n for which the equations (4.26)-(4.44) are valid. For example, if $c_{0,n} \geq c$, when $n=0,1,\ldots,N'<N-1$ and $c_{0,n}<c$, when $n=N'+1,\ldots,N-1$, equation (4.27) holds only for $n=1,2,\ldots,N'$. Furthermore, there is a change also in (4.28)-(4.34) as well as in the boundary and initial conditions. Modifications of this type require a new formulation and solution of the model.

In the following, we consider the derivation of the reliability characteristic R_0 only in two particular cases, as examples of the general procedure. We consider $R_0(\hat{c},t)$ and $R_0(C,t)$, the extreme values of $R_0(c,t)$ (for definition of $\hat{c}$ see equation (4.141)). To get the model for $R_0(\hat{c},t)$ we need modifications of the first type only, whereas for the model for $R_o(C,t)$ both types of modifications are needed.

In the modified model for $R_0(\hat{c},t)$ only states 5-14 are absorbing, states 2-4 remaining unchanged for all values of $1=1,2,\ldots,L$ and $n=1,2,\ldots,N-1$ (and state 1, of course, remaining non-absorbing). Using the appropriate modifications of type 1, the model gets the form of the equations (4.26)-(4.44), with the following exceptions

(i) in (4.26) we have $\gamma(x)\equiv 0$ and $\alpha_k(x)\equiv 0$, $k=1,2,\ldots,K$,

(ii) in (4.29) and (4.31) we have $\varphi(x)\equiv 0$,

(iii) in (4.30) and (4.32) we have $\gamma(x)\equiv 0$,

(iv) in (4.33) and (4.34) we have $\alpha_k(x)\equiv 0$, $k=1,2,\ldots,K$,

(v) in (4.35) and (4.36) we have $\alpha_k(x)\equiv 0$, $k=1,2,\ldots,K$, $\gamma(x)\equiv 0$, and $\varphi(x)\equiv 0$ for $n=N-1$.

The solving of the reliability model now proceeds in the same way as that of the availability model in paragraph 4.2. After applying Laplace transforms and discrete transforms we finally achieve a system of (L+1)MN linear equations, corresponding to equations (4.83), (4.89) and (4.90) in paragraph 4.2, and in these equations we have (L+1)MN unknowns to determine, i.e. unknown functions corresponding to the functions in (4.81). Making the "absorbtion substitutions" (i)-(v) above in the equations (4.26)-(4.44), we get, instead of equations (4.83), (4.89) and (4.90), the following equations for determining the unknown functions (4.81)

$$(4.147)\quad \begin{cases} (s + \eta + \lambda + i\mu + N\nu)\,\overline{A}_i(s) - \overline{f}(s+\eta+\lambda+i\mu+[N-1]\nu) \times \\ \overline{B}_{0,i,N-1}(0,s) - \sum_{l=1}^{L} \overline{b}_l(s+\eta+i\mu+N\nu)\overline{B}_{1,i,N}(0,s) = \binom{M}{i}, \\ \qquad i=1,2,\ldots,M, \end{cases}$$

$$(4.148)\quad \begin{cases} \delta_{n,1}N\nu \sum_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m}\overline{A}_i(s) - \sum_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m} \times \\ \sum_{j=N-n}^{N-1} (-1)^{j-N+n}\binom{j}{N-n}\overline{B}_{0,i,j}(0,s) \\ + (1-\delta_{n,N-1}) \sum_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m} \sum_{j=N-n-1}^{N-1} (-1)^{j-N+n+1} \times \\ \binom{j}{N-n-1}\overline{f}(s+\eta+\lambda+i\mu+j\nu)\overline{B}_{0,i,j}(0,s) \\ + \sum_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m} \sum_{j=N-n}^{N} (-1)^{j-N+n}\binom{j}{N-n} \times \\ \sum_{l=1}^{L} \overline{b}_l(s+\eta+i\mu+j\nu)\overline{B}_{1,i,j}(0,s) = 0, \quad m=0,1,\ldots,M-1, \\ \qquad n=1,2,\ldots,N-1, \end{cases}$$

$$(4.149)\quad \begin{cases} \delta_{n,0}\lambda_l \sum_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m}\overline{A}_i(s) + (1-\delta_{n,0})\lambda_l \times \\ \sum_{i=M-m}^{M} (-1)^{i-M+m}\binom{i}{M-m} \sum_{j=N-n}^{N-1} (-1)^{j-N+n}\binom{j}{N-n} \times \\ \dfrac{1-\overline{f}(s+\eta+\lambda+i\mu+j\nu)}{s+\eta+\lambda+i\mu+j\nu}\,\overline{B}_{0,i,j}(0,s) - \sum_{i=M-m}^{M} (-1)^{i-M+m} \times \\ \binom{i}{M-m} \sum_{j=N-n}^{N} (-1)^{j-N+n}\binom{j}{N-n}\overline{B}_{1,i,j}(0,s) = 0, \\ l=1,2,\ldots,L,\; m=0,1,\ldots,M-1,\; n=1,2,\ldots,N-1. \end{cases}$$

The value of the reliability characteristic $R_0(c,t)$ when $c = \hat{c}$ (the minimum positive level of performance of the system) is now the probability that at time t the system is operable, either with full capacity or at a level of reduced performance:

$$(4.150)\quad R_0(\hat{c},t) = \sum_{l=0}^{L}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} P_{0,l,m,n}(t).$$

In order to get the expression for the quantity $R_0(\hat{c},t)$ into the form of the system's parameters we first derive its Laplace transform. The Laplace transform of $R_0(\hat{c},t)$ is

$$(4.151)\quad \overline{R}_0(\hat{c},s) = \sum_{l=0}^{L}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \overline{P}_{0,l,m,n}(s),$$

where the $\overline{P}_{0,l,m,n}(s)$'s get their values from equations (4.91), (4.92) and (4.93), however in such a way that the functions $\overline{A}_i(s)$, $\overline{B}_{0,i,j}(0,s)$ and $\overline{B}_{1,i,j}(0,s)$ are now solved from equations (4.147) - (4.149).

In the model for $R_0(C,t)$, i.e. the particular value of $R_0(c,t)$ when $c = C$, the only state that remains unchanged is state 1; also all the states indicating any reduction in the level of performance become absorbing. Thus

$$(4.152)\quad R_0(C,t) = \sum_{m=0}^{M-1} P_{0,0,m,0}(t).$$

Now the model for $P_{0,0,m,0}(t)$, $m=0,1,\ldots,M-1$, simply becomes

$$(4.153)\quad \begin{cases} [d/dt + \eta + \lambda + (M-m)\mu + N\nu]P_{0,0,m,0}(t) = \\ (1-\delta_{m,0})\mu(M-m+1)P_{0,0,m-1,0}(t), \quad m=0,1,\ldots,M-1 \end{cases}$$

with the initial condition

$$(4.154)\quad P_{0,0,m,0}(0) = \delta_{m,0}, \quad m=0,1,\ldots,M-1.$$

The Laplace transform of equation (4.153), with the initial condition (4.154), is

$$(4.155)\quad \begin{cases} [s + \eta + \lambda + (M-m)\mu + N\nu]\overline{P}_{0,0,m,0}(s) = \delta_{m,0} + \\ (1-\delta_{m,0})\mu(M-m+1)\overline{P}_{0,0,m-1,0}(s), \quad m=0,1,\ldots,M-1, \end{cases}$$

which after application of the discrete transform (4.67) becomes

(4.156) $(s + \eta + \lambda + i\mu + N\nu)\overline{A}_i(s) = \binom{M}{i}, \quad i=1,2,\dots,M.$

Equation (4.156) gives for $\overline{A}_i(s)$

(4.157) $\overline{A}_i(s) = \binom{M}{i}(s + \eta + \lambda + i\mu + N\nu)^{-1}, \quad i=1,2,\dots,M.$

From (4.157) we further obtain

(4.158) $A_i(t) = \binom{M}{i} e^{-(\eta+\lambda+i\mu+N\nu)t}, \quad i=1,2,\dots,M.$

On the other hand, as inverse Laplace transforms of equation (4.138), we have

(4.159) $\sum_{m=0}^{M-1} P_{0,0,m,0}(t) = \sum_{i=1}^{M} (-1)^{i-1} A_i(t).$

Combining the results of equations (4.152), (4.158) and (4.159) we finally obtain for $R_0(C,t)$

(4.160) $R_0(C,t) = e^{-(\eta+\lambda+N\nu)t} \sum_{i=1}^{M} (-1)^{i-1}\binom{M}{i} e^{-i\mu t}.$

The values $\hat{c}$ and C are the extreme values for the possible (positive) levels of performance of the system. The two cases above, the derivations of the values for reliability of levels of performance, where c has the values $\hat{c}$ and C respectively, have been considered only as examples of the general derivation of $R_0(c,t)$. With any other value of c ($\hat{c} < c < C$) the derivation proceeds along similar lines, the number of modifications in equations (4.26) - (4.44) only being different with different values of c.[1]

If one again wants to remain within the bounds of the traditional concepts of reliability, this is also possible by using $R_0(c,t)$. In order to get R(t), the reliability of the two-stage operable or inoperable system at time t (the probability that the system is operable during the whole interval from 0 to t), we select a value for c corresponding to the level of performance below which the system is regarded as inoperable, and at and above which it is regarded as operable, and determine $R_0(c,t)$.

1 We need to consider, of course, only the values corresponding to the different possible performance levels of the system.

In $R_0(\hat{c},t)$ and $R_0(C,t)$ we thus have two separate values for the traditional reliability R(t), based on different interpretations of the concept of failure. $R_0(\hat{c},t)$ gives the reliability when only total inoperability is regarded as system failure and $R_0(C,t)$ when also all the failures which merely reduce the level of performance are regarded as system failures.

4314 Mean operation time of levels of performance before failure

The fourth generalized reliability characteristic defined in Chapter 3 was T_0, or the mean operation time of levels of performance before failure. The value of the function T_0 at a level of performance c ($0 < c \leq C$) was defined as the expected value of that time period after which the level of performance of the system for the first time becomes less than c. The derivation of $T_0(c)$ is quite simple after the reliability of levels of performance has been determined. We have namely

(4.161) $T_0(c) = \int_0^\infty R_0(c,t)dt$

or, using the Laplace transform of $R_0(c,t)$,

(4.162) $T_0(c) = \lim_{s \to 0} \overline{R}_0(c,s).$[1]

It will be seen from equation (4.162) that the expression for $T_0(c)$ can be derived also in the case of general repair time distributions, without any exact knowledge of the types of these distributions. This is because $\overline{R}_0(c,s)$, and therefore also $T_0(c)$, can be determined when the system has general distributions, whereas, for example, $R_0(c,t)$, the inversion of $\overline{R}_0(c,s)$, can only be determined with fixed distributions.

As a particular case for $T_0(c)$ we get, by integration from equation (4.160), $T_0(C)$, i.e. the expected value for the time when the level of performance of the system for the first time drops below the level of full capacity

1 $\int_0^\infty f(t)dt = \lim_{s \to 0} \overline{f}(s)$, see SPIEGEL (1965), p. 7.

$$(4.163)\qquad T_0(C) = \int_0^{\infty} R_0(C,t) = \sum_{i=1}^{M} (-1)^{i-1}\binom{M}{i}(\eta+\lambda+i\mu+N\nu)^{-1}.$$

The same result can be obtained by using (4.162), the Laplace transforms of (4.152) and (4.159), and (4.157)

$$(4.164)\qquad \begin{cases} T_0(C) = \lim\limits_{s\to 0} \sum\limits_{m=0}^{M-1} \bar P_{0,0,m,0}(s) = \lim\limits_{s\to 0} \sum\limits_{i=1}^{M} (-1)^{i-1}\bar A_i(s) \\ \qquad = \lim\limits_{s\to 0} \sum\limits_{i=1}^{M} (-1)^{i-1}\binom{M}{i}(s+\eta+\lambda+i\mu+N\nu)^{-1} \\ \qquad = \sum\limits_{i=1}^{M} (-1)^{i-1}\binom{M}{i}(\eta+\lambda+i\mu+N\nu)^{-1}. \end{cases}$$

432 On the steady state reliability of the system

In section 4244 we considered the steady state or asymptotic behaviour of the system by determining the steady state values of the state probabilities. In Chapter 3 and also earlier in this chapter we have noted that the characteristics of the extended concepts of reliability can be derived from the state probabilities. Of course this is true also under the steady state, when the steady state probabilities (4.103) to (4.113) can be used.

The only relevant characteristics for the steady state reliability of a failable system are the availability characteristics, i.e. the availability of levels of performance (A_0) and the mean availability of the capacity (A_C). The mean operation time of levels of performance before failure (T_0) is not a function of time at all, and for the reliability of levels of performance (R_0) we have

$$(4.165)\qquad \lim_{t\to\infty} R_0(c,t) = 0, \text{ for all } c,\ 0 < c \le C,$$

i.e. with probability 1 the system becomes, sooner or later, totally inoperable. In the following, we derive the steady state values for the two generalized availability characteristics. We use the notation

$$(4.166)\qquad \tilde A_0(c) = \lim_{t\to\infty} A_0(c,t),\ 0 < c \le C,$$

$$(4.167)\qquad \tilde A_C = \lim_{t\to\infty} A_C(t).$$

Applying (4.10) to (4.139) and (4.144), we get

$$(4.168)\qquad \begin{cases} \tilde A_0(c) = \sum\limits_{i=1}^{M} (-1)^{i-1}\tilde A_i + \sum\limits_{i=1}^{M} (-1)^{i-1} \sum\limits_{c_{0,n}\ge c} \sum\limits_{j=N-n}^{N-1} (-1)^{j-N+n} \times \\ \binom{j}{N-n} \dfrac{1-\bar F(\eta+\lambda+i\mu+j\nu)}{\eta+\lambda+i\mu+j\nu}\tilde B_{0,i,j} \\ + \sum\limits_{i=1}^{M} (-1)^{i-1} \sum\limits_{c_{1,0}\ge c} \dfrac{1-\bar b_1(\eta+i\mu+N\nu)}{\eta+i\mu+N\nu}\tilde B_{1,i,N} \\ + \sum\limits_{i=1}^{M} (-1)^{i-1} \sum\limits_{c_{1,n}\ge c} \sum\limits_{j=N-n}^{N} (-1)^{j-N+n}\binom{j}{N-n} \times \\ \dfrac{1-\bar b_1(\eta+i\mu+j\nu)}{\eta+i\mu+j\nu}\tilde B_{1,i,j}, \end{cases}$$

$$(4.169)\qquad \begin{cases} \tilde A_C = \sum\limits_{i=1}^{M} (-1)^{i-1}\tilde A_i + \sum\limits_{i=1}^{M} (-1)^{i-1} \sum\limits_{n=1}^{N-1} w_{0,n} \sum\limits_{j=N-n}^{N-1} (-1)^{j-N+n} \times \\ \binom{j}{N-n} \dfrac{1-\bar F(\eta+\lambda+i\mu+j\nu)}{\eta+\lambda+i\mu+j\nu}\tilde B_{0,i,j} \\ + \sum\limits_{i=1}^{M} (-1)^{i-1} \sum\limits_{n=0}^{N-1} \sum\limits_{l=1}^{L} w_{l,n} \sum\limits_{j=N-n}^{N} (-1)^{j-N+n}\binom{j}{N-n} \times \\ \dfrac{1-\bar b_l(\eta+i\mu+j\nu)}{\eta+i\mu+j\nu}\tilde B_{l,i,j}, \end{cases}$$

where $\tilde A_i$, $\tilde B_{0,i,j}$ and $\tilde B_{1,i,j}$ are obtained from equations (4.114) - (4.116).

5 ON THE POTENTIAL USE OF THE MODEL

51 Reliability analysis of a given system

In Chapter 4 we formulated and solved a reliability model for the system under study. As a matter of fact we had to formulate and solve a number of models, one for the availability characteristics A_0 and A_C, and one for each different possible value of the performance level of the system, in order to obtain the reliability and mean operation time characteristics R_0 and T_0, respectively. At the same time we presented the fundamental and most important way of using the model(s). In paragraph 4.3, namely, we used the model(s) in the reliability analysis of the system under study by determining the different reliability characteristics of the system.

This is also generally the first and basic way of utilizing the model: to determine either a particular or all the most important reliability characteristics of a particular system, i.e. of a system where the structure, the operational properties of the components, and the repair policy are fixed. In this way it will be possible to use the model for studying the operational behaviour and reliability of the system under different circumstances, e.g. under a different repair policy, under a different type of redundancy in subsystem S_3 etc. (preserving, however, the general structural and operational framework of the system).

The aim of this study was to formulate and solve the reliability model of the system and use it to determine the characteristics of extended concepts of reliability. As we have just pointed out, that aim has now been achieved. The following paragraph contains some general remarks on the potential use of model. These remarks may lead to ideas for further research in this subject.

52 Dependence of system reliability upon the properties and control of the system

521 Redundancy among the components

As will be remembered from the general description, redundancy was introduced into the system. It was assumed that subsystem S_3 consisted of M identical components which were redundantly connected in parallel. Other types of redundancy would also have been possible, however. For example, in subsystem S_3 we might have had

(a) stand-by redundancy:[1] out of M redundantly connected components only one component takes part in the operation and the rest are kept as stand-by arrangements; the system is assumed to switch over to the second component automatically when the first component fails; the system fails only when all the stand-by connected components fail,

(b) lightly loaded stand-by:[2] the same as stand-by redundancy with the exception that a stand-by component may also fail when it is not operating though the probability of this is less than the probability of failure when it is operating,

(c) m-out-of-M structure:[3] all M components start operating together as soon as the system is put into operation (as in redundancy in parallel); the system is operable if and only if at least m components ($1 \leq m \leq M$) are operable.

Different types of redundancy in S_3 produce different reliability values for the system. By carrying out a reliability analy-

1 PIERUSCHKA (1963), p. 76.

2 GNEDENKO et al. (1969), p. 303; the names used in GNEDENKO et al. for parallel redundancy and stand-by redundancy are loaded stand-bys and unloaded stand-bys, respectively.

3 BARLOW and PROSCHAN (1965), p. 216.

sis of the system under different types of redundancy we can discover the effects the type of redundancy has upon the reliability properties of the system. Although in this analysis we need different models for different cases, the models are so similar to that in Chapter 4 that they can be formulated and solved according to the same principles and with the same methods as in the case of redundancy in parallel.

An increase in the number of redundant components in S_3 also increases the reliability of the system. The amount of this increase can be measured with the model, by obtaining the values of the reliability characteristics for different values of M. A redundancy optimization with respect to the number of redundant components is also possible, but this presupposes knowledge not only of the cost of additional redundant components but also of the cost incurred as a result of the unreliability of subsystem S_3.

It was assumed that subsystem S_4 was composed of N independent components operating in parallel branches or flow lines. Each branch covers one Nth part of the capacity of the system. An increase in the number of components in S_4 makes the system more liable to fail, but the consequences of the failures of the S_4-components become less serious (the extent of the reduction in level of performance due to the failure of a particular component decreases). The dependence of system reliability on the number of components in S_4 thus presents another new and interesting reliability problem.

522 Operability and repairability of a single component

As well as by introducing redundancy, the reliability of a system can also be increased by decreasing the failure intensity of the components and by decreasing their repair times, i.e. by increasing the operability and repairability of the components. Measures available for this purpose are for example programmes of preventive maintenance, regular inspection and replacement policies etc., even the use of more reliable components.

One possible way of using the reliability model would be to study the effects of the above-mentioned measures upon the reliability of the system. In the model, the programmes of preventive maintenance etc. are seen as changes in failure and repair time distributions, either as changes in the values of the parameters or as changes in the types of distributions. The consequences of these changes are then reflected in the solutions of the model and in the reliability of the system. In the model we assumed exponential failure time distributions and general repair time distributions. Keeping these assumptions valid (which enables us to use the results of the model already formulated and solved) we are able to study, with regard to failure time distributions, the effects of changes in the failure rates of the components, and with regard to repair time distributions, the effects of changes both in the parameter values and in the types of distributions.

523 Repair policy for the system

The general policy for repair of the system was assumed to be the preemptive repeat repair discipline (see section 411). Instead of this discipline we could have introduced some other appropriate policy. Further use of the model becomes now possible when an alternative repair policy is employed. Every policy requires a model of its own, but these models will be so similar that they can be dealt with according to the same principles and with the same methods. Among others, the following repair policies are common in reliability literature[1]

(a) head of line repair discipline: the repair policy is "first come first served", which means that a failed component from any priority class, if taken for repair, will be repaired first and only after this will other failed component (with the highest existing priority) be dealt with,

1 See GOVIL and KUMAR (1970) and NATARAJAN (1968).

(b) preemptive resume repair discipline: in this policy, a failed component from a higher priority class is taken for repair immediately, i.e. the component preempts the repair facility even if the facility was already engaged in the repair of a component from a lower priority class; when the latter component is taken back by the repair facility, the repair starts from the point where it was left off.

In addition to the general repair policy (preemptive repeat repair, head of line repair, preemptive resume repair etc.) there are other points in the management of repairs which are worth discussing. One such point is the repair of subsystem S_3. In the repair of S_3 it was assumed that a "minimum repair policy" is followed: i.e. the failed components of S_3 are not repaired until all M components have failed; in the case of such a total failure, all the failed components are repaired together. Another possible policy in the repair of subsystem S_3 would be a course of action which includes the provision of opportunistic repairs:[1] along with the repair of a failed component from a higher priority class (e.g. a failed S_1-component) opportunistic repair of all the failed S_3-components is also carried out.

1 Cf. KULSHRESTHA (1968a).

REFERENCES

von ALVEN, W.H. (ed.), Reliability Engineering, Englewood Cliffs (N.J.) 1964

BARLOW, R.E., PROSCHAN, F., Mathematical Theory of Reliability, New York 1965

COX, D.R., The Analysis of Non-Markovian Stochastic Processes by the Inclusion of Supplementary Variables, Proceedings of the Cambridge Philosophical Society, Vol. 51 (1955), pp. 433-441

DAS, P., Point-Wise Availability of an Electronic System with Reduced Efficiency Class of Components, Microelectronics and Reliability, Vol. 10 (1971), No. 2, pp. 61-66

GARG, R.C., Dependability of a Complex System Having Two Types of Components, IEEE Transactions on Reliability, Vol. R-12 (1963), No. 3, pp. 11-16

GARG, R.C., Dependability of a Complex System with General Waiting Time Distributions, IEEE Transactions on Reliability, Vol. R-12 (1963), No. 3, pp. 17-23

GNEDENKO, B.V., BELYAYEV, Y.K., SOLOVYEV, A.D., Mathematical Methods of Reliability Theory, New York 1969

GOVIL, A.K., Dependability of a Complex System with General Repair Distribution under Preemptive Priority Resume, Zeitschrift für angewandte Mathematik und Mechanik, Band 50 (1970), Heft 8, pp. 499-500

GOVIL, A.K., Complex System Point-Wise Availability with Priority Repair, Computing, Vol. 7 (1971), pp. 275-280

GOVIL, A.K., Stochastic Behaviour of a Complex System with Bulk Failure and Priority Repairs, Revue Francaise d'Informatique et de Recherche Opérationnelle, Vol. 5, (1971), V-1, pp. 51-56

GOVIL, A.K., Priority Effect on Pointwise Availability of the System, Revue Francaise d'Automatique, Informatique et Recherche Operationnelle, Vol. 6 (1972), V-1, pp. 47-56

GOVIL, A.K., Operational Behaviour of a Complex System Having Shelf-Life of the Components under Preemptive Resume Repair Discipline, Microelectronics and Reliability, Vol. 13 (1974), No. 2, pp. 97-101

GOVIL, A.K., KUMAR, S., Stochastic Behaviour of a Complex System under Priority Repair, Computing, Vol. 6 (1970), pp. 200-213

GUPTA, P.P., Complex System Reliability with General Repair Time Distributions Under Preemptive Repeat Repair Discipline, Microelectronics and Reliability, Vol. 12 (1973), pp. 145-150

GUPTA, P.P., Complex System Reliability with Exponential Repair Time Distributions under Head-of-Line Repair Discipline, Microelectronics and Reliability, Vol. 12 (1973), pp. 151-158

HONKO, J., Liiketaloustiede (Business Economics), Espoo 1974

HRABÁK, J., Pohotovost a provozuschopnost energetických zařizeni (Availability and Operability of Power Plant Equipment), Energetika, Vol. 25 (1975), No. 2, pp. 76-78

HRABÁK, J., Ukazatele pohotovosti a provozuschopnosti elektrizačni soustavy PEV a jejich výpočet (Characteristics of Availability and Operability of an Electrification System of Thermal Power Plants and Their Calculation), Energetika, Vol. 25 (1975), No. 4, pp. 168-171

JORGENSON, D.W., McCALL, J.J., RADNER, R., Optimal Replacement Policy, Amsterdam 1967

KAPUR, P.K., KAPOOR, K.R., Reliability Model with 1 or 2 Repair Channels, IEEE Transactions on Reliability, Vol. R-24 (1975), No. 2, pp. 147-148

KEILSON, J., KOOHARIAN, A., On Time Dependent Queuing Processes, Annals of Mathematical Statistics, Vol. 31 (1960), pp. 104-112

KODAMA, M., Probabilistic Analysis of a Multicomponent Series-Parallel System under Preemptive Repeat Repair Discipline, Operations Research, Vol. 24 (1976), No. 3, pp. 500-515

KULSHRESTHA, D.K., Analytical Study of a Stand-by Redundant Complex System with General Repair Time Distribution, Opsearch, Vol. 3 (1966), pp. 139-150

KULSHRESTHA, D.K., Reliability of a Parallel Redundant Complex System, Operations Research, Vol. 16 (1967), No. 1, pp. 28-35

KULSHRESTHA, D.K., Operational Behaviour of a Multicomponent System Having Stand-by Redundancy with Opportunistic Repairs, Unternehmensforschung, Band 12 (1968), Heft 3, pp. 159-172

KULSHRESTHA, D.K., Operational Behaviour of a Complex System, Unternehmensforschung, Band 12 (1968), Heft 4, pp. 232-241

KULSHRESTHA, D.K., Application of Discrete Transforms in Reliability of a Parallel Redundant System, Metrika, Vol. 18 (1972). Fasc. 2/3, pp. 145-153

LINDELÖF, E., Differentiali- ja integralilasku ja sen sovellutukset II (Differential and Integral Calculus and Applications II), Helsinki 1932

LLOYD, D.K., LIPOW, M., Reliability: Management, Methods and Mathematics, Englewood Cliffs (N.J.) 1962

NAKAMICHI, H., FUKUTA, J., TAKAMATSU, S., KODAMA, M., Reliability Consideration on a Repairable Multicomponent System with Redundancy in Parallel, Journal of the Operations Research Society of Japan, Vol. 17 (1974), No. 1, pp. 39-48

NATARAJAN, R., Long-Run Availability of Paralleled Systems, Metrika, Vol. 13 (1968), Fasc. 2/3, pp. 104-122

PIERUSCHKA, E., Principles of Reliability, Englewood Cliffs (N.J.) 1963

POLOVKO, A.M., Fundamentals of Reliability Theory, New York 1968

RAU, J.G., Optimization and Probability in Systems Engineering New York 1970

SPIEGEL, M.R., Theory and Problems of Laplace Transforms, New York 1965

SRINIVASAN, V.S., First Emptiness in the Spare Parts Problem for Repairable Components, Operations Research, Vol. 16 (1968), No. 2, pp. 407-415

SRIVASTAVA, S.S., GARG, R.C., GOVIL, A.K., Stochastic Behaviour of an Intermittently Working System with Stand-by Redundancy, Microelectronics and Reliability, Vol. 10 (1971), No. 3, pp. 159-167

SRIVASTAVA, S.S., GARG, R.C., GOVIL, A.K., Behaviour of a Complex System with Stand-by Redundancy under Different Repair Echelons, Microelectronics and Reliability Vol. 10 (1971), No. 5, pp. 375-380

SRIVASTAVA, S.S., GARG, R.C., MALHOTRA, N.K., KUMAR, S., Stochastic Behaviour of a Parallel Redundant Complex System with Three Types of Components, Computing, Vol. 7 (1971), No. 1-2, pp. 83-90

TACAĆS, L., Introduction to the Theory of Queues, New York 1962

THIRUVENGADAM, K., JAISWAL, N.K., Application of Discrete Transforms to a Queuing Process of Servicing Machines, Opsearch, Vol. 1 (1964), No. 2

VARMA, G.K., Operational Behaviour of a Complex System with Two out of M Failed Components, Microelectronics and Reliability, Vol. 12 (1973), No. 2, pp. 107-110

VIRTANEN, I., Simulation Model for Maintenance, Conference Papers of NordDATA-72, pp. 410-431, Helsinki 1972

VIRTANEN, I., Kunnossapidon simulointimalli (Simulation Model for Maintenance), not published, available in the Library of The Turku School of Economics, Turku 1972

VIRTANEN, I., Simulation Model for Optimizing the Maintenance Policy of a Pulp and Paper Mill, Congress Papers of the 2nd European Maintenance Congress, pp. 191-221, Paris 1974

VIRTANEN, I., Monikomponenttisen stokastisesti vioittuvan systeemin käyttäytymisestä ja luotettavuudesta (On Operational Behaviour and Reliability of a Multicomponent Stochastically Failing System), Publications of The Turku School of Economics, Series AI-4:1974. Turku 1974

VIRTANEN, I., Application of Supplementary Variables and Discrete Transforms in Availability and Reliability Analysis of a Parallel Redundant System with General Failure and Repair Time Distributions, Publications of The Turku School of Economics, Series A-12:1976, Turku 1976

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