Lecture Notes in
Control and
Information Sciences

Edited by A.V. Balakrishnan and M. Thoma

IFIP

38

System Modeling and Optimization

Proceedings of the 10th IFIP Conference
New York City, USA, August 31 – September 4, 1981

Edited by R.F. Drenick and F. Kozin

Springer-Verlag
Berlin · Heidelberg · New York

OPTIMAL MAINTENANCE POLICY AND SALE DATE FOR A MACHINE WITH RANDOM DETERIORATION AND SUBJECT TO RANDOM CATASTROPHIC FAILURE

Ilkka Virtanen
University of Vaasa
School of Business Studies
Vaasa, Finland

Abstract. The problem of providing optimal maintenance for a machine during its service life and simultaneously selecting its optimal sale date is considered from a control-theoretic viewpoint. Both the deterioration and the life time of the machine are considered as random processes. The stochastic maximum principle is applied to derive the conditions for the optimal maintenance policy and for the optimal planned sale date which maximize the expected net present value of the machine. An explicit solution is found analytically for the problem in the special case when some of the random processes of the model are independent of time and thus simply random variables. The case of one particular life-time distribution, the exponential case, is analyzed in full detail.

1 Introduction

When a machine is used for production purposes and it ages, it suffers one of the two fates - either there is a gradual deterioration or a sudden failure. The first situation means more frequent repairs, a decrease in performance of the machine etc., the machine produces decreasing net receipts over time. This deterioration can be partially offset via preventive maintenance, and there also exists, of course, the possibility of selling the machine at any time, although its salvage value declines over time. The second situation makes the machine unusable for production and it has to be junked and replaced by a new machine.

Since Näslund [7] had initiated the control theory approach to the problem of simultaneous maintenance and sale date optimization, Thompson [10] first formulated for the problem an explicit model and solved it in detail. Thompson's model is deterministic: the machine cannot fail and its deterioration obeys a given mathematical law. Other formulations for the deterministic problem have been later presented e.g. by Arora and Lele [3], Bensoussan et al. [5] and Scott and Jefferson [9]. Kamien and Schwartz [6] developed a stochastic model where the failure part of the problem was included but the degradation of the machine with age was not considered. Due to Alam and Sarma [2] is a model where both these features have been incorporated in a single model. The deterioration is taken as deterministic, whereas the machine is subject to random failure. Also the author of this paper has recently presented a generalized model [11] for the problem considered in [2].

In this paper we consider a model where the random nature of both the deterioration and the life time of the machine are taken into account. Alam et al. [1] have earlier presented a model where both these aspects have been included, but that model contains the unsatisfactory aspect that the sale date of the machine is not as an object of optimization: the machine is kept as long as it is operable, even though its use would not be profitable any more. This may lead to an unprofitable use of the machine and to an inproper optimum for the problem. Therefore, we generalize model [1] and make it more realistic by taking also the sale date of the machine (called the planned sale date due to the possibility of machine failing before that time) as a tool of optimization.

2 Review of the models of Thompson and Alam et al.

Thompson considers the following problem: find the optimal maintenance policy $u(t)$ and the optimal sale date T for a machine to maximize the present value $V(T)$ of the machine given by

$$(1) \qquad V(T) = S(T)\exp(-rT) + \int_0^T [pS(t) - u(t)]\exp(-rt)dt$$

where the salvage value $S(t)$ is affected by the deterioration factor and the amount and the effectiveness of preventive maintenance according to the differential equation

$$(2) \qquad \frac{dS(t)}{dt} = -\delta(t) + f(t)u(t) , \quad S(0) = S_0 .$$

In (1) and (2) r is the discount rate, $\delta(t)$ the deterioration rate, $f(t)$ the maintenance effectiveness function, and p is the (constant) production rate. The maintenance function $u(t)$ is the control variable satisfying the requirement

$$(3) \qquad 0 \le u(t) \le U , \quad 0 \le t \le T ,$$

and $V(t)$ and $S(t)$ are the state variables. The solution of the problem can be found by a direct application of the maximum principle (see [10], pp. 545-547).

Alam et al. [1] take Thompson's model as the starting point and they model deterioration as a random process whereas machine failing is not considered in the first phase. We call this model Alam I. The deterioration rate is described by the following stochastic differential equation

$$(4) \qquad d\underline{\delta}(t)/dt = \underline{\alpha}(t) - \beta(t)u(t)$$

with the stochastic boundary condition

(5) $\underline{\delta}(0) = \underline{\delta}_0$.

The stochastic processes $\underline{\alpha}(t)$ and $\underline{\delta}(t)$ as well as the random variable $\underline{\delta}_0$ are assumed to be defined on a certain sample space Ω, the probability measure joining with Ω being P (generally speaking, we use the notation $\underline{z}$ or $\underline{z}(t)$ to indicate that the quantity z or z(t) is a random variable or a stochastic process, respectively). Because of (1) and (2), also the salvage value S(t) and the present value V(T) will now be stochastic processes on Ω, denoted by $\underline{S}(t)$ and $\underline{V}(T)$, respectively.

The problem (Alam I) is now to choose $u^*(t)$ and T so as to maximize

(6) $\bar{V}(T) = E\{\underline{V}(T)\} = \int_\Omega \underline{V}(T)dP$, where

(7) $\underline{V}(T) = \underline{S}(T)\exp(-rT) + \int_0^T [\underline{p}(t)\underline{S}(t) - u(t)]\exp(-rt)dt$

subject to the state equations

(8) $d\underline{S}(t)/dt = -\underline{\delta}(t) + f(t)u(t)$, $0 \leq t \leq T$; $\underline{S}(0) = S_0$

and (4) with boundary condition (5), and to the control constraint (3).

Applying the stochastic maximum principle, the solution of the problem can be derived. An analytic solution is possible in the special case when $\underline{\alpha}(t)$ and $\underline{p}(t)$ don't depend on time, they are simply random variables: $\underline{\alpha}(t) \equiv \underline{\alpha}$ and $\underline{p}(t) \equiv \underline{p}$ (for the solution of problem Alam I see [1], pp. 1073-1074).

In the second phase Alam et al. [1] take also the probability of machine failure into account and derive now the optimal maintenance policy for the machine, whereas the sale date of the machine is not considered, but the machine is assumed to be kept as long as it is operable. We call this model Alam II. Let τ denote the random life time of the machine and let $p_\tau(t;u(s), 0 \leq s \leq t)$, $P_\tau(t;u(s), 0 \leq s \leq t)$ and $Q_\tau(t;u(s), 0 \leq s \leq t)$ denote its density function, cumulative distribution function and reliability function, respectively. Further, let $p_\tau(t;u)$, $P_\tau(t;u)$ and $Q_\tau(t;u)$ compactly represent these quantities. Assuming the deterioration and failure processes mutually independent, the following model (Alam II) can be stated: choose an optimal policy $u^*(t)$ so as to satisfy the state equations (4), (5) and (8) and the control constraint (3) and to maximize the expectation

(9) $E_P\{E_\tau[\underline{V}(\tau)]\} = E_P\{\underline{V}_F\} = \int_\Omega \underline{V}_F dP$, where

(10) $\underline{V}_F = E_\tau[\underline{V}(\tau)] = \int_0^\infty \{[Q_\tau(t;u)\underline{p}(t) + p_\tau(t;u)]\underline{S}(t) - Q_\tau(t;u)u(t)\}\exp(-rt)dt$

is the expectation of $\underline{V}(\tau)$, the quantity in (7) with T considered as the random variable τ, and the expectation being taken with respect to τ. The second expectation E_P in (9) is with respect to the probability measure P.

Again, the solution of the problem can be found via application of the stochastic maximum principle. An analytic solution becomes possible when $\underline{\alpha}(t)$ and $\underline{p}(t)$ are simply random variables: $\underline{\alpha}(t) \equiv \underline{\alpha}$ and $\underline{p}(t) = \underline{p}$, and when failure probability is independent of maintenance: $p_\tau(t;u) = p_\tau(t)$ and, hence, $Q_\tau(t;u) = Q_\tau(t)$ (see [1], pp. 1076-1077).

3 The generalized model

Both the models Alam I and Alam II contain deficiencies in their formulation. The former represents an unrealistic situation in practice by assuming the machine as unbreakable, the latter may lead to an unprofitable use of the machine by forcing the owner to use the machine until it fails, regardless of its ever declining quality and productivity. In this paper, we provide for the problem a generalized formulation in which the above disadvantages are not included. We seek a planned sale date T and a planned maintenance policy $u^*(t)$, $0 \leq t \leq T$, for machine until it is sold or it fails and must be junked, whichever comes first, so as to maximize the expected present value of the machine. The machine is assumed to suffer random deterioration as well as to be subject to random catastrophic failure.

The state equations considered are again (8) and (4) with (5). The control constraint is (3). In the expressions above T now denotes the planned sale date of the machine, i.e. T is the time at which the machine will be sold provided it has not failed and been junked before that time.

The present value of the machine at time t is, provided the machine is still operable, according to (7),

$$(11) \qquad \underline{V}(t) = \underline{S}(t)\exp(-rt) + \int_0^t [\underline{p}(t)\underline{S}(t) - u(t)]\exp(-rt)dt \ .$$

Let $\underline{V}_0(T)$ denote the present value which will be really obtained when the planned sale date of the machine is T. By assuming the junk value of the machine equal to its salvage value at the failure time, we get

$$(12) \qquad \underline{V}_0(T) = \begin{cases} \underline{V}(T)\ , & \text{if } \tau > T \\ \underline{V}(\tau)\ , & \text{if } \tau < T\ . \end{cases}$$

Now, taking the expectation of $\underline{V}_0(T)$ with respect to the random variable τ and assuming mutual independence between the deterioration and failure processes, we get

$$(13) \quad \underline{V}_F(T) = E_\tau\{\underline{V}_0(T)\}$$
$$= \int_0^T \underline{V}(t)p_\tau(t;u)dt + \int_T^\infty \underline{V}(T)p_\tau(t;u)dt$$
$$= \int_0^T \underline{V}(t)p_\tau(t;u)dt + Q_\tau(T;u)\underline{V}(T) \ .$$

Substituting (11) in (13) we get after some labour (for details, see [12])

$$(14) \quad \underline{V}_F(T) = Q_\tau(T;u)\underline{S}(T)\exp(-rT)$$
$$+ \int_0^T \{[\underline{p}(t)Q_\tau(t;u) + p_\tau(t;u)]\underline{S}(t) - Q_\tau(t;u)u(t)\}\exp(-rt)dt \ .$$

Our problem is now to choose a (planned) optimal maintenance policy $u^*(t)$ and a (planned) optimal sale date T so as to maximize the expectation

$$(15) \quad \bar{V}_F(T) = E_P\{\underline{V}_F(T)\} = \int_\Omega \underline{V}_F(T)dP$$

where $\underline{V}_F(T)$ is given by (14) and the expectation is taken with respect to the probability measure P over the sample space Ω.

We can readily see that our generalized model is of the same form as the model Alam I, only with the coefficients of $\underline{S}(t)$ and $u(t)$ modified. The generalized model coincides with the model Alam I, when we only set $p_\tau(t;u) \equiv 0$ (the failure part of the model is omitted). We can also see that our generalized model coincides with the model Alam II, if we in (14) set $T = \infty$ to give (10) (the sale date optimization is omitted). Our model thus contains both the models Alam I and Alam II as its special cases.

4 Solution by stochastic maximum principle

The solution of the problem needs an application of the stochastic maximum principle. For this we must assume certain smoothness and regularity conditions: f, β, and u are piecewise continuous, $\underline{\alpha}(t)$ and $\underline{p}(t)$ and, hence, $\underline{\delta}(t)$, $\underline{S}(t)$ and $\underline{V}_F(t)$ are stochastic processes and $\underline{\delta}_0$ a random variable on a sample space Ω which is assumed to be a compact subset of an Euclidean space. The stochastic processes and the random variable are random quantities with respect to the probability measure P on Ω (for a detailed and strict description of the assumptions for the stochastic maximum principle see [4], pp. 876-878).

To solve the problem, we first form the Hamiltonian, which is now a random variable

$$(16)\quad \underline{H} = H(\underline{S}, \underline{\delta}, u, \underline{\lambda}_1, \underline{\lambda}_2, t)$$
$$= -\{[\underline{p}(t)Q_\tau(t;u) + p_\tau(t;u)]\underline{S}(t) - Q_\tau(t;u)u(t)\}\exp(-rt)$$
$$+ \underline{\lambda}_1(t)[-\underline{\delta}(t) + f(t)u(t)] + \underline{\lambda}_2(t)[\underline{\alpha}(t) - \beta(t)u(t)] ,$$

where the adjoint variables $\underline{\lambda}_1(t)$ and $\underline{\lambda}_2(t)$ also are stochastic processes on Ω and satisfy the stochastic differential equations

$$(17)\quad \begin{cases} \dfrac{d\underline{\lambda}_1(t)}{dt} = -\dfrac{\partial \underline{H}}{\partial \underline{S}} = [\underline{p}(t)Q_\tau(t;u) + p_\tau(t;u)]\exp(-rt) \\[2ex] \dfrac{d\underline{\lambda}_2(t)}{dt} = -\dfrac{\partial \underline{H}}{\partial \underline{\delta}} = \underline{\lambda}_1(t) \end{cases}$$

with the boundary conditions

$$(18)\quad \begin{cases} \underline{\lambda}_1(T) = -\dfrac{\partial}{\partial \underline{S}}[Q_\tau(T;u)\underline{S}(T)\exp(-rT)] = -Q_\tau(T;u)\exp(-rT) \\[2ex] \underline{\lambda}_2(T) = -\dfrac{\partial}{\partial \underline{\delta}}[Q_\tau(T;u)\underline{S}(T)\exp(-rT)] = 0 . \end{cases}$$

To find the solution for our problem we should proceed as follows. First we consider T as fixed and apply the stochastic maximum principle (i.e. minimize $E_p\{H\}$ with respect to u, see [4], pp. 879-880) to obtain the optimal maintenance policy $u^*(t)$ for $0 \leq t \leq T$. Then we choose T so as to maximize $\overline{V}_F(T)$.

There exist, however, two reasons, why an analytic solution for this general case is not possible, and, in order to find out the solution, we had to use one of the iterative computational techniques. First, equations (17) are general stochastic differential equations, and secondly, the failure probability $p_\tau(t;u)$ depends on the maintenance performed. Here we present the solution for the problem in the special case where an analytic solution is possible. We make the following additional assumptions. First we assume that $\underline{\alpha}(t)$ and $\underline{p}(t)$ are independent of t, $\underline{\alpha}(t) \equiv \underline{\alpha}$ and $p(t) \equiv \underline{p}$ are simply random variables. The assumption makes it possible to obtain an explicit solution for the co-state equations (17) with (18) and, hence, for the state equations. The solution is achieved by replacing the required stochastic quantities with their expected values. The second assumption is that the failure probability is independent of maintenance: $p_\tau(t;u) = p_\tau(t)$ and, hence $Q_\tau(t;u) = Q_\tau(t)$. With this assumption, an analytic application of the stochastic maximum principle is possible.

Applying the stochastic maximum principle, i.e. minimizing $E_p\{\underline{H}\}$ with respect to u for the above special problem, the following condition for the optimal maintenance policy $u^*(t)$ is obtained

$$(19)\qquad u^*(t) = \begin{cases} U, & \text{if } E_p\{G(Q_\tau, \underline{\lambda}_1, \underline{\lambda}_2, t)\} < 0 \\ \text{arbitrary} \in [0,U], & \text{if } E_p\{G(Q_\tau, \underline{\lambda}_1, \underline{\lambda}_2, t)\} = 0 \\ 0, & \text{if } E_p\{G(Q_\tau, \underline{\lambda}_1, \underline{\lambda}_2, t)\} > 0 . \end{cases}$$

In (19) we have denoted

$$(20)\qquad G(Q_\tau, \underline{\lambda}_1, \underline{\lambda}_2, t) = Q_\tau(t)\exp(-rt) + \underline{\lambda}_1(t)f(t) - \underline{\lambda}_2(t)\beta(t) .$$

Equation (19) shows that the optimal maintenance policy is bang-bang. The possible switching point(s) T', where the level of maintenance is changed from U to 0 or vice versa, satisfy the switching equation $E_p\{G(Q_\tau, \underline{\lambda}_1, \underline{\lambda}_2, T')\} = 0$ or

$$(21)\qquad f(T') = [\beta(T')\overline{\lambda}_2(T') - Q_\tau(T')\exp(-rT')]/\overline{\lambda}_1(T')$$

In (21), $\overline{\lambda}_1(t)$ and $\overline{\lambda}_2(t)$ denote the expectations of $\underline{\lambda}_1(t)$ and $\underline{\lambda}_2(t)$, respectively, the expectations being with respect to the probability measure P.

Thus far we have considered the planned sale date T as fixed. We still have to choose T so as to maximize the expected present value $\overline{V}_F(T)$. Using similar reasoning as Thompson ([10], p. 546), and assuming mutual independence between the random variables $\underline{\alpha}$ and $\underline{p}$ we get the following condition for the optimal planned sale date T:

$$(22)\qquad \overline{S}(T) = \{\overline{\delta}(T) - [f(T)-1]u^*(T)\}/(\overline{p}-r) .$$

5 A particular case: exponentially distributed life time

We shall now demonstrate an explicit calculation of the optimal maintenance policy (19) for a machine with exponentially distributed life time. Therefore, let $p_\tau(t) = \sigma\exp(-\sigma t)$ and, hence, $Q_\tau(t) = \exp(-\sigma t)$ (for $t \geq 0$). As it is well known, the parameter of the distribution ($=\sigma$) corresponds to the constant failure rate of the machine.

By using the above expressions for $p_\tau(t)$ and $Q_\tau(t)$ the optimal maintenance policy (19) becomes

$$(23)\qquad u^*(t) = \begin{cases} U, & \text{if } \overline{G}(t) < 0 \\ \text{arbitrary} \in [0,U], & \text{if } \overline{G}(t) = 0 \\ 0, & \text{if } \overline{G}(t) > 0, \end{cases}$$

where

$$\begin{aligned}(24)\qquad \overline{G}(t) &= E_p\{G(Q_T,\underline{\lambda}_1,\underline{\lambda}_2,t)\} = G(Q_T,\overline{\lambda}_1,\overline{\lambda}_2,t) \\ &= \exp\{-(r+\sigma)t\}\Big\{(r+\sigma) - f(t)\Big[(\overline{p}+\sigma) - (\overline{p}-r)\exp\{-(r+\sigma)(T-t)\}\Big] \\ &\quad - \beta(t)\Big[(\overline{p}+\sigma)[1 - \exp -(r+\sigma)(T-t)\}]/(r+\sigma) \\ &\quad - (\overline{p}-r)\exp\{-(r+\sigma)(T-t)\}(T-t)\Big]\Big\}/(r+\sigma)\ .\end{aligned}$$

The bang-bang optimal policy (23) may have none, one or more switching points. For a switching point T' we have $\overline{G}(T') = 0$.

In section 3 we showed that our generalized model contains the prior models Alam I and Alam II as its special cases. As we now in the exponential case have obtained an explicit solution for the problem, we can also compare the results.

Comparing the optimal maintenance policy (23) with the optimal policy of the model Alam I (see eg. (10) in [1]), we see that they are of the same form. If we instead of the discount rate r in the model Alam I use the 'risk-adjusted' discount rate $r+\sigma$ and instead of the mean production rate $\overline{p}$ use the 'risk-adjusted' mean production rate $\overline{p}+\sigma$, we get (23). Or on the contrary, if we in our model ignore the possibility of random failure and set $\sigma = 0$, the two models coincide. The failure rate σ may thus be interpreted as a risk premium which is to be used to adjust both the discount rate and the mean production rate to the level of those in a certainty-equivalent problem.

In the model Alam II, instead of optimizing also the sale date of the machine, the machine was assumed to be kept until it fails and becomes junked, or in our terms, the planned sale date was fixed to infinity. Setting $T = \infty$ in (24) we obtain, that (23) coincides with the optimal policy for the model Alam II (see eg. (29) in [1]).

In this paper we have pointed out the importance of the sale date optimization also in the case of random machine life. The effect of this optimization can very clearly be demonstrated by a numerical example (see [12], pp. 17-19).

References

[1] Alam, M., Lynn, J.W. and Sarma, V.V.S., "Optimal maintenance policy for equipment subject to random deterioration and random failure. A modern control theory approach", Int. J. Systems Sci., 1976, Vol. 7, No. 9, 1971-1080

[2] Alam, M. and Sarma, V.V.S., "Optimum maintenance policy for an equipment subject to deterioration and random failure", IEEE Trans. Syst. Man Cybernet., 1974, Vol. SMC-4, No. 2, 172-175

[3] Arora, S.R. and Lele, P.T., "A note on optimal maintenance policy and sale date of a machine", Mgmt Sci., 1970, Vol. 17, No. 3, 170-173

[4] Baum, R.F., "Optimal control systems with stochastic boundary conditions and state equations", Ops Res., 1972, Vol. 20, 875-887

[5] Bensoussan, A., Hurst, E.G. Jr. and Näslund, B., Management applications of modern control theory, Amsterdam 1974

[6] Kamien, M.I. and Schwartz, N.L., "Optimal maintenance and sale age for a machine subject to failure", Mgmt Sci., 1971, Vol. 17B, No. 8, 495-504

[7] Näslund, B., "Simultaneous determination of optimal repair policy and service life", Swedish J. Econ., 1966, Vol. 68, No. 2, 63-73

[8] Saaty, T.L., Modern nonlinear equations, New York 1967

[9] Scott, C.H. and Jefferson, T.R., "A bilinear control model for optimal maintenance", Int. J. Control, 1979, Vol. 30, No. 2, 323-330

[10] Thompson, G.L., "Optimal maintenance policy and sale date of a machine", Mgmt Sci., 1968, Vol. 14, No. 9, 543-550

[11] Virtanen, I., "Optimal maintenance policy and planned sale date for a machine subject to deterioration and random failure", European J. Operational Res. (to appear)

[12] Virtanen, I., "Optimal maintenance policy and sale date for a machine with random deterioration and subject to random catastrophic failure", Lappeenranta University of Technology, Dept. of Physics and Mathematics, Report 2/1980