Ilkka Viitanen

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Ilkka Virtanen, Lic.Ph.
The Turku School of Economics

On Operational Behaviour and Reliability of a Complex System; an Application of Supplementary Variables and Laplace Transforms

1. Introduction

Reliability of a system with several independent components can be brought up to a given level in two main ways: it is made sure that the components themselves are so reliable that the goal laid down for the whole system is achieved, or alternatively, in the case of low-reliability components, the lack of reliability in these components is eliminated through suitable measures of system technics. The most important measure of this kind is with no doubt to introduce redundancy among the strategic components.

Redundancy of components means additional costs, usually even to a considerable extent. Because of this it is important to have methods for an accurate measuring of that improvement in reliability which has been obtained as a counterbalance of these costs. The purpose of this paper is to present a method based on supplementary variable technique and Laplace transforms that makes it possible to consider mathematically the operation of a quite general stochastic system with redundancy in one of its strategic parts. On the grounds of the results describing the system's operational behaviour several conclusions about the reliability of the system are drawn.

2. Description of the system

The real system under consideration is composed of two subsystems, designated as S_1 ja S_2, such that subsystem S_1 consists of M identical components which are redundantly connected while S_2 contains N independent different types of components connected in series. In order to function satisfactorily the system must have both of its components operable: subsystems S_1 and S_2 are themselves in series. Redundancy in S_1 is in parallel: all the components of S_1 start operating as soon as the system is put into operation.

The repair of any failed component is possible only when the whole system stops operating. The operation of the system is not, however, purposely stopped for repair, repairs are not carried out until the system has got into a standstill as a consequence of a failure. This means the failure of some component in S_2 or a complete failure in S_1. About the nature and influence of faults it is assumed that contributions (cost, repair time etc.) necessary for restoring a failed component in S_1 are considerable higher than those for a component in S_2 or for a used, still operable component in S_1.

The specific repair policy to be considered in this connection obeys the following principles. After a complete failure in S_1 all the failed components in S_1 are repaired, the components in S_2 are left unattended. In the case of a failure in S_2 the failed component is repaired; in addition to this the operable components in S_1 are pre-serviced, whereupon they can be regarded as new. Failed components in S_1 are left because of their labouriousness to wait for a general overhaul of subsystem S_1.

3. Assumptions in the model and general definition of the problem

In the paper the operation of a system, which is adapted to the general framework of the last section, is considered in the form of a mathematical model. Several factors having an effect on the behaviour of the system are stochastic by nature. In the model these stochastic elements are treated as the following random variables: time between failures of a component, repair time of a component in S_2 or of the whole subsystem S_1, and waiting time elapsing from failure to the beginning of repair. The following assumptions concerning the distributions of the random variables are made.

For the greatest part of the system's random variables only one requirement has been imposed. The probability density of random variables must exist on the region of the whole non-negative real axis. The shape of the probability density and so the type of the distribution can otherwise be thorougly arbitrary. A general distribution of this kind is introduced into the model with the help of its probability density $f(x)$ or "intensity function" $r(x)$, that is in connection with $f(x)$ in the form of equation[1]

$$(3.1) \qquad f(x) = r(x) \exp\left\{ - \int_0^x r(x)\, dx \right\}.$$

For example, in the case of the random variable time between failures f(x) is called failure density and r(x) failure (or hazard) rate. The following random variables have general distributions in the model: time between failures for a S_1-component (each component has the same distribution), repair time and waiting time for the whole subsystem S_1, and repair and waiting times for components in S_2 (each component has distributions of its own). Distributions of the time between failures for the S_2-components are, however, supposed to be exponential, i.e. the failure rates are assumed as constants.

Reliability properties of a system with the general structure described in section 2 have been before considered by KULSHRESTHA[2]. In this paper the model developed by Kulshrestha has been considerably generalized and enlarged. Not only the actual repair time but the time that elapses in waiting for repair has been introduced into the model as well. The waiting time is caused by such factors as spare part supply, lack of any vacant repair crew etc. Further the use of only the exponential distribution as a model for the time between failures for components in S_1 has been replaced by the general distribution that covers all the continued distributions.

The inclusion of general distributions into the model increases its potential properties and utilization possibilities in a considerable degree. The model can naturally serve as a usual computing algoritm in order to solve problems concerning the reliability of a spesific system when the system's distributions are known or have been estimated. First of all the model is, however, a general method that makes it possible to consider reliability properties of systems of a certain type. The methodologic character of the model is very clearly revealed

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1. About the functions r(x) definition, interpretation and connections with probability density and distribution functions see e.g. Barlow and Proschon, p. 10

2. Kulshrestha, D.K.: Reliability of a Repairable Multicomponent system with Redundancy in Parallel

when the steady-state behaviour of the system is considered. It can namely be shown that after a sufficiently long time has gone since the operation started the reliability properties of the system cease to depend on the type of most of the system's distributions. For computing the steady-state reliability of the system it is sufficient to know only the mean values of these distributions.

Formulation of the model and search for solutions are for their key parts based on supplementary variable and Laplace transform techniques. The former has turned out to be a very efficient method in constructing state equations for originally non-Markovian stochastic processes. The method has been developed on the region of the general theory of stochastic processes[1]. From there it has got its way out through queuing applications[2] to reliability theory and has found a remarkable application area there[3]. When the supplementary variable technique is used, the state equations of the system become partial differential equations, in the case under consideration the state equations are, due to the special properties of the system, differential equations, too. Solutions for the state equations are derived with the help of Laplace transforms and they are got into a closed form for both the transient state and steady state. Existence of the steady state and expressions for steady-state solutions are derived using certain limit properties of the Laplace transform. Noteworthy in this connection is that the steady-state solutions can be derived without inverse Laplace transforms. Besides, only the expected values for most of the system's distributions must be known, the nature of the distributions has otherwise no effect on the solutions.

1. Cox, p. 443
2. Keilson and Kooharian, p. 104
3. As first applications in this area can be mentioned e.g. Garg's and Kulshrestha's papers

4. Mathematical model for the system

4.1. Notations

Let us as a first step in the formulation of the model specify the states of the system and introduce some notations. We can note that, at any instant of time, the system is found in one of the states listed below:

m: the system is operable, m out of M components in S_1 have failed, $m = 0,\dots, M-1$

W: the system is waiting for repair due to failure of all the components in S_1

R: the system is under repair of all the M components in S_1

w_{mi}: the system is waiting for repair due to failure of the ith component in S_2; m out of M components in S_1 have failed, $m=0,\dots,M-1$; $i=1,\dots,N$

r_{mi}: the system is under repair due to failure of the ith component in S_2; m out of M components in S_1 have failed, the M-m operable components in S_1 are going through the pre-service, $m=0\dots,M-1$; $i=1,\dots,N$

Figure 1 shows the states of the system and the one-step transitions among them.

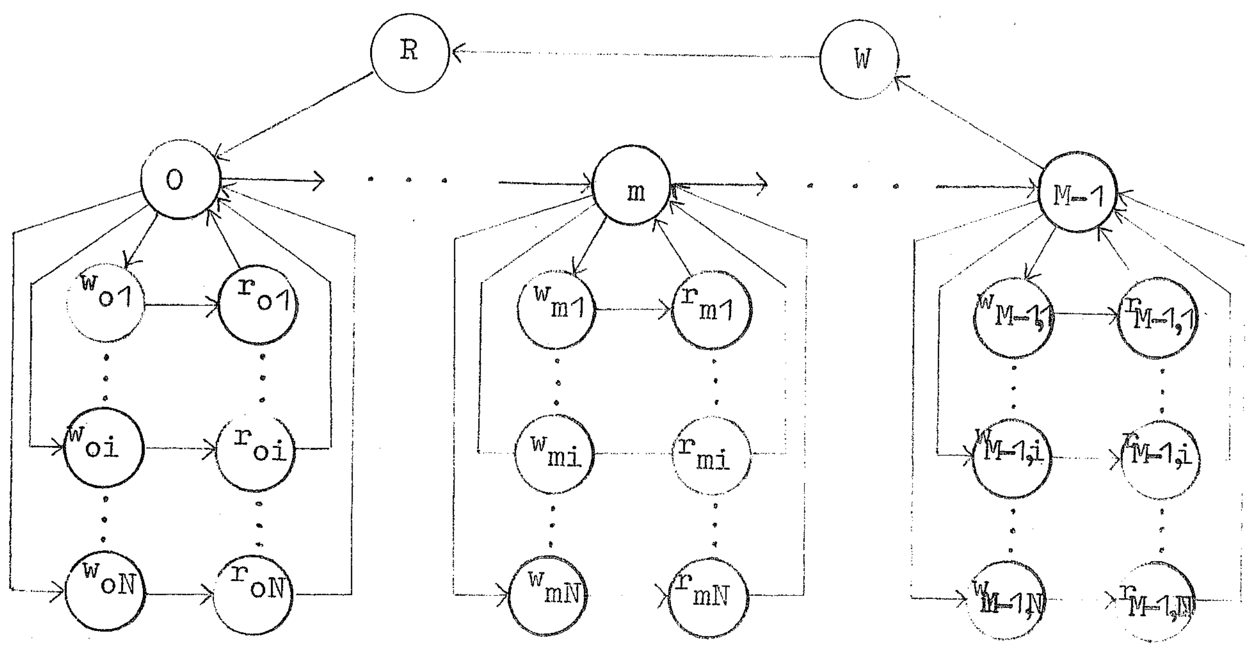

Fig. 1. The state transition diagram for the system

Denote,

$\sum$: set of the states of the system

S: subscript for a state, $S \in \sum$

$P_S(x,t)$: probability density for the joint probability that at time t the system is in state S and the elapsed time since it came into that state is x (just the variable x is the supplementary variable appearing in the name of the method to be used)

$P_S(t)$: probability that the system at time t is in state S; evidently $P_S(t) = \int_0^\infty P_S(x,t)\, dx$

$\alpha(x)$: failure rate for components in S_1

$A(x)$: probability density for the time between failures of a S_1-component; $A(x) = \alpha(x) \exp\left\{-\int_0^x \alpha(x)\, dx\right\}$

$\beta(x)$: repair rate for S_1

$B(x)$: probability density for S_1's repair time distribution, $B(x) = \beta(x) \exp\left\{-\int_0^x \beta(x)\, dx\right\}$

$\gamma(x)$: waiting rate for S_1

$C(x)$: probability density for S_1's waiting time distribution; $C(x) = \gamma(x) \exp\left\{-\int_0^x \gamma(x)\, dx\right\}$

λ_i: constant failure rate for the ith component in S_2, $i=1,\ldots,N$; $\sum_{i=1}^{N} \lambda_i = \lambda$

$\eta_i(x)$: waiting rate for the ith component in S_2, $i=1,\ldots,N$

$H_i(x)$: probability density for the waiting time distribution of the ith component in S_2; $H_i(x)=\eta_i(x)\exp\left\{-\int_0^x \eta_i(x)dx\right\}$, $i=1,\ldots,N$

$\mu_i(x)$: repair rate for the ith component in S_2, $i=1,\ldots,N$

$M_i(x)$: probability density for the repair time distribution of the ith component in S_2; $M_i(x)= \mu_i(x)\exp\left\{-\int_0^x \mu_i(x)dx\right\}$, $i=1,\ldots,N$

$A_k(x) = k\alpha(x) \exp\left\{-\int_0^x k\alpha(x)\, dx\right\}$, $k=1,\ldots,M$

The expected value of a random variable is denoted by the symbol '$-$'; thus the mean repair time for subsystem S_1, for instance, is denoted by $\overline{B}$. Other notations will be explained when they for the first time appear.

4.2. Formulation of the model

The system has to be found at any time instant t in one of the states specified in section 4.1. A transition from one state to another takes place according to figure 1 and is governed by the probability laws defined above. Consideration of the system's behaviour during a time interval (t, t+$\triangle$) leads to the following forward difference equations:

(4.1) $$P_m(x+\triangle,t+\triangle)=P_m(x,t)\left\{\left[1-\alpha(x)\triangle\right]^{M-m}\prod_{i=1}^{N}(1-\lambda_i\triangle)\right\}+o(\triangle),$$

$m=0,\ldots,M-1$

(4.2) $$P_W(x+\triangle,t+\triangle)= P_W(x,t)\left[1-\gamma(x)\triangle\right]+o(\triangle)$$

(4.3) $$P_R(x+\triangle,t+\triangle)= P_R(x,t)\left[1-\beta(x)\triangle\right]+o(\triangle)$$

(4.4) $$P_{wmi}(x+\triangle,t+\triangle)= P_{wmi}(x,t)\left[1-\eta_i(x)\triangle\right]+o(\triangle)$$

(4.5) $$P_{rmi}(x+\triangle,t+\triangle)= P_{rmi}(x,t)\left[1-\mu_i(x)\triangle\right]+o(\triangle)$$

$m=0,\ldots,M-1;\ i=1,\ldots,N$

As $\triangle \to 0$, equations (4.1) to (4.5) result in the following set of variable coefficient partial difference equations:

(4.6) $$\left[\partial/\partial x+\partial/\partial t+(M-m)\alpha(x)+\lambda\right]P_m(x,t)=0,\quad m=0,\ldots,M-1$$

(4.7) $$\left[\partial/\partial x+\partial/\partial t+\gamma(x)\right]P_W(x,t)=0$$

(4.8) $$\left[\partial/\partial x+\partial/\partial t+\beta(x)\right]P_R(x,t)=0$$

(4.9) $$\left[\partial/\partial x+\partial/\partial t+\eta_i(x)\right]P_{wmi}(x,t)=0$$

(4.10) $$\left[\partial/\partial x+\partial/\partial t+\mu_i(x)\right]P_{rmi}(x,t)=0$$

$m=0,\ldots,M-1$; $i=1,\ldots,N$

Variables x and t are time quantities so that the density functions in equations (4.6) to (4.10) has been defined only on the region

(4.11) $$\Gamma=\left\{(x,t)\ \middle|\ x\geq 0,\ t\geq 0\right\},$$

The equations are, therefore, to be solved under boundary conditions on the boundaries

(4.12) $$\Gamma_x=\left\{(x,0)\ \middle|\ x\geq 0\right\},\quad \Gamma_t=\left\{(0,t)\ \middle|\ t\geq 0\right\}$$

of Γ.

The conditions on boundary Γ_t desribe the system's transition from one state to another and they are

$$(4.13)\quad P_o(0,t) = {}_0\int^{\infty} P_R(x,t)\beta(x)dx + \sum_{i=1}^{N} {}_0\int^{\infty} P_{roi}(x,t)\mu_i(x)dx$$

$$(4.14)\quad P_m(o,t) = {}_0\int^{\infty} P_{m-1}(x,t)(M-m+1)\alpha(x)dx + \sum_{i=1}^{N} {}_0\int^{\infty} P_{rmi}(x,t)\mu_i(x)dx, \quad m=1,\ldots,M-1$$

$$(4.15)\quad P_W(0,t) = {}_0\int^{\infty} P_{M-1}(x,t)\alpha(x)dx$$

$$(4.16)\quad P_R(0,t) = {}_0\int^{\infty} P_W(x,t)\gamma(x)dx$$

$$\left.\begin{array}{ll}(4.17) & P_{wmi}(0,t) = {}_0\int^{\infty} P_m(x,t)\lambda_i dx = \lambda_i P_m(t) \\ (4.18) & P_{rmi}(0,t) = {}_0\int^{\infty} P_{wmi}(x,t)\eta_i(x)dx\end{array}\right\} \begin{array}{l} m=0,\ldots,M-1; \\ i=1,\ldots,N\end{array}$$

Assuming the system to start initially in state 0, we obtain the following initial conditions (the conditions on boundary Γ_x):

$$(4.19)\quad P_m(x,0) = \delta_{mo}\delta(x), \quad m=0,\ldots,M-1$$

$$(4.20)\quad P_W(x,0) = P_R(x,0) = P_{wmi}(x,0) = P_{rmi}(x,0) = 0, \quad m=0,\ldots,M-1;\ i=1,\ldots,N$$

where δ_{mo} is the Kronecker delta and $\delta(x)$ the Dirac delta function.

Equations (4.6) to (4.10) and (4.13) to (4.20) provide a complete description of the state of the system at any instant of time. Below they are called the state equations of the system.

4.3. Solutions

Let the Laplace transform of a function F(t) be denoted by f(s), i.e.,

$$(4.21)\quad \mathcal{L}\left\{F(t)\right\} = {}_0\int^{\infty} e^{-st}F(t)dt = f(s)$$

Applying the Laplace transform to the set of equations (4.6) to (4.10) and employing initial conditions (4.19) to (4.20), we obtain

$$(4.22)\quad \left[\partial/\partial x + s + (M-m)\alpha(x) + \lambda\right]p_m(x,s) = \delta_{mo}\delta(x), \quad m=0,\ldots,M-1$$

$$(4.23)\quad \left[\partial/\partial x + s + \gamma(x)\right]p_W(x,s) = 0$$

$$(4.24)\quad \left[\partial/\partial x + s + \beta(x)\right]p_R(x,s) = 0$$

$$\left.\begin{array}{ll}(4.25) & \left[\partial/\partial x + s + \eta_i(x)\right]p_{wmi}(x,s) = 0 \\ (4.26) & \left[\partial/\partial x + s + \mu_i(x)\right]p_{rmi}(x,s) = 0\end{array}\right\} m=0,\ldots,M-1; i=1,\ldots,N$$

The Laplace transforms of boundary conditions (4.13) to (4.18) are

$$(4.27)\quad p_o(0,s) = \int_0^\infty p_R(x,s)\beta(x)dx + \sum_{i=1}^{N} \int_0^\infty p_{roi}(x,s)\mu_i(x)dx$$

$$(4.28)\quad p_m(0,s) = \int_0^\infty p_{m-1}(x,s)(M-m+1)\alpha(x)dx + \sum_{i=1}^{N} \int_0^\infty p_{rmi}(x,s)\mu_i(x)dx, \quad m=1,\dots,M-1$$

$$(4.29)\quad p_W(0,s) = \int_0^\infty p_{M-1}(x,s)\alpha(x)dx$$

$$(4.30)\quad p_R(0,s) = \int_0^\infty p_W(x,s)\gamma(x)dx$$

$$\left.\begin{aligned}(4.31)\quad & p_{wmi}(0,s) = \lambda_i p_m(s)\\ (4.32)\quad & p_{rmi}(0,s) = \int_0^\infty p_{wmi}(x,s)\eta_i(x)ds\end{aligned}\right\} m=0,\dots,M-1;\ i=1,\dots,N$$

State probabilities $P_S(t)$ and probability densities $P_S(x,t)$, $S\in\Sigma$, are related to each others like shown in section 4.1. Applying the Laplace transform to this relation, we get

$$(4.33)\quad p_S(s) = \int_0^\infty p_S(x,s)dx, \quad S\in\Sigma .$$

On integration over interval (0,x) equations (4.22) to (4.26) give

$$(4.34)\quad p_m(x,s) = \left[\delta_{mo} + p_m(0,s)\right] \exp\left\{-(s+\lambda)x - \int_0^x (M-m)\alpha(x)dx\right\}, \quad m=0,\dots,M-1$$

$$(4.35)\quad p_W(x,s) = p_W(0,s)\ \exp\left\{-\ sx - \int_0^x \gamma(x)dx\right\}$$

$$(4.36)\quad p_R(x,s) = p_R(0,s)\ \exp\left\{-\ sx - \int_0^x \beta(x)dx\right\}$$

$$\left.\begin{aligned}(4.37)\quad & p_{wmi}(x,s) = p_{wmi}(0,s)\ \exp\left\{-\ sx - \int_0^x \eta_i(x)dx\right\}\\ (4.38)\quad & p_{rmi}(x,s) = p_{rmi}(0,s)\ \exp\left\{-\ sx - \int_0^x \mu_i(x)dx\right\}\end{aligned}\right\} m=0,\dots,M-1;\ i=1,\dots,N$$

The expression for $p_m(0,s)$, $m=0,\dots,M-1$, in equation (4.34) is derived starting in (4.33). Substituting the value of $p_m(x,s)$ from (4.34) into (4.33) and integrating, we first obtain

$$\begin{aligned}p_m(s) &= \int_0^\infty p_m(x,s)dx\\ &= \left[\delta_{mo} + p_m(0,s)\right] \int_0^\infty \exp\left\{-(s+\lambda)x - \int_0^x (M-m)\alpha(x)dx\right\}dx\\ (4.39)\qquad &= \frac{\delta_{mo}+p_m(0,s)}{s+\lambda}\left[1 - \int_0^\infty e^{-(s+\lambda)x}(M-m)\alpha(x)\exp\left\{-\int_0^x (M-m)\alpha(x)dx\right\}dx\right]\\ &= \left[\delta_{mo} + p_m(0,s)\right]\frac{1 - a_{M-m}(s+\lambda)}{s+\lambda}, \quad m=0,\dots,M-1\end{aligned}$$

so that the expression for $p_m(0,s)$ becomes

$$(4.40) \quad p_m(0,s) = \frac{s+\lambda}{1-a_{M-m}(s+\lambda)} \, p_m(s) - \delta_{m0}, \quad m=0,\ldots,M-1$$

Integrating over $(0,\infty)$, applying equations (4.29) to (4.32) and (4.35) to (4.38) repeatedly, and bearing further result (4.40) and form (3.1) of the probability density in mind, we get the Laplace transforms of the state probabilities into a form, where their dependence on (so far unknown) quantities $p_m(s)$ is revealed:

$$(4.41) \quad p_W(s) = \frac{(s+\lambda)\, a(s+\lambda)\left[1-c(s)\right]}{s\left[1-s(s+\lambda)\right]} \, p_{M-1}(s)$$

$$(4.42) \quad p_R(s) = \frac{(s+\lambda)\, a(s+\lambda)\left[1-b(s)\right]c(s)}{s\left[1-a(s+\lambda)\right]} \, p_{M-1}(s)$$

$$\left.\begin{aligned} (4.43) \quad p_{wmi}(s) &= \frac{\lambda_i\left[1-h_i(s)\right]}{s}\, p_m(s) \\ (4.44) \quad p_{rmi}(s) &= \frac{\lambda_i\, h_i(s)\left[1-m_i(s)\right]}{s}\, p_m(s) \end{aligned}\right\} \quad m=0,\ldots,M-1;\, i=1,\ldots,N$$

To determine the expressions for $p_m(s)$ we still have equations (4.27) and (4.28) at our disposal. On relevant substitutions and simplification these equations result in the variable coefficient difference equations

$$(4.45) \quad \left\{ \begin{aligned} &\left[\frac{s+\lambda}{1-a_{M-m}(s+\lambda)} - \sum_{i=1}^{N} \lambda_i\, h_i(s)\, m_i(s)\right] p_m(s) - \\ &\frac{(s+\lambda)\, a_{M-m+1}(s+\lambda)}{1-a_{M-m+1}(s+\lambda)}\, p_{m-1}(s) = 0, \quad m=1,\ldots,M-1. \end{aligned}\right.$$

As solutions for these equations we have[1]

$$(4.46) \quad p_m(s) = \frac{1-a_{M-m}(s+\lambda)}{(s+\lambda)\, a_{M-m}(s+\lambda)} \; \frac{Q(s,M-m-1)}{Q(s,M) - b(s)\,c(s)}, \quad m=0,\ldots,M-1$$

where notation $Q(s,k)$ is used to mean expression

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1. see Virtanen, pp. 46–48

$$(4.47)\quad \begin{cases} Q(s,k) = \prod_{r=1}^{k} \dfrac{s+\lambda - \left[1 - a_r(s+\lambda)\right] \sum_{i=1}^{N} \lambda_i h_i(s) m_i(s)}{(s+\lambda)\, a_r(s+\lambda)}, & k \geq 1 \\ Q(s,0) = 1 \end{cases}$$

After that the Laplace transforms for all the state probabilities are known, they can be got from equations (4.41) to (4.44) and (4.46) to (4.47). It may be noted that for all values of s holds[1]

$$(4.48)\quad \sum_{m=0}^{M-1} p_m(s) + p_W(s) + p_R(s) + \sum_{m=0}^{M-1} \sum_{i=1}^{N} \left[p_{wmi}(s) + p_{rmi}(s)\right] = \frac{1}{s}.$$

From the point of view of the state probabilities result (4.48) means that, for all values of t, equation

$$(4.49)\quad \sum_{m=0}^{M-1} P_m(t) + P_W(t) + P_R(t) + \sum_{m=0}^{M-1} \sum_{i=1}^{N} \left[P_{wmi}(t) + P_{rmi}(t)\right] = 1$$

holds. This points out the fact that the set of the system's states is well defined: the state probabilities are all mutually exclusive and totally exhaustive probabilities.

Now for given values of the system's probability densities the relations (4.41) to (4.44) and (4.46) can be inverted to give the desired state probabilities as their inverse Laplace transforms

$$(4.50)\quad P_S(t) = \mathcal{L}^{-1}\left\{p_S(s)\right\}, \quad S \in \Sigma .$$

Final form of the state probabilities so remains to depend on the specific properties of the distributions of the system and must be in each case separately cleared up.

4.4. Behaviour under steady state

From the transient-state solutions we can see that the state probabilities have under this stage a (naturally) strong dependence on the distributions of the system. In the following it is shown, however, that after the system has been in operation a sufficiently long time it gets into a steady state, where e.g. the types of repair and waiting time distributions cease to have an effect on the values of the state probabilities. To get the

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1. see Virtanen, pp. 49–51

system finally into a steady state presumes only a few additional assumptions for the system's distributions: the expected values for repair and waiting times must exist. The expressions for steady-state probabilities can be brought into their final solved form in the case of general distributions, too. In addition, the results are found without inverse Laplace transforms which often turn out to be very inconvenient to carry out. Applying the well known result in Laplace transform[1], viz

(4.51) $$\lim_{t\to\infty} F(t) = \lim_{s\to 0} s\, f(s)$$

to the set of equations (4.41) to (4.46), we obtain the following steady-state probabilities (where notation P_S is used for the limit $\lim_{t\to\infty} P_S(t)$, $S \in \sum$):

(4.52) $$P_m = \frac{1-a_{M-m}(\lambda)}{\lambda a_{M-m}(\lambda)} \; \frac{S_M(\lambda)}{1+\sum_{i=1}^{N}\lambda_i(\overline{H}_i+\overline{M}_i)+S_M(\lambda)(\overline{B}+\overline{C})}, \quad m=0,\ldots,M-1$$

(4.53) $$P_W = \frac{S_M(\lambda)\overline{C}}{1+\sum_{i=1}^{N}\lambda_i(\overline{H}_i+\overline{M}_i)+S_M(\lambda)(\overline{B}+\overline{C})}$$

(4.54) $$P_R = \frac{S_M(\lambda)\overline{B}}{1+\sum_{i=1}^{N}\lambda_i(\overline{H}_i+\overline{M}_i)+S_M(\lambda)(\overline{B}+\overline{C})}$$

(4.55) $$P_{wmi} = \frac{1-a_{M-m}(\lambda)}{\lambda a_{M-m}(\lambda)} \; \frac{S_M(\lambda)\,\lambda_i\overline{H}_i}{1+\sum_{i=1}^{N}\lambda_i(\overline{H}_i+\overline{M}_i)+S_M(\lambda)(\overline{B}+\overline{C})}$$

(4.56) $$P_{rmi} = \frac{1-a_{M-m}(\lambda)}{\lambda a_{M-m}(\lambda)} \; \frac{S_M(\lambda)\,\lambda_i\overline{M}_i}{1+\sum_{i=1}^{N}\lambda_i(\overline{H}_i+\overline{M}_i)+S_M(\lambda)(\overline{B}+\overline{C})}$$

$m=0,\ldots,M-1;\; i=1,\ldots,N$

where

(4.57) $$S_M(\lambda)=\left\{\sum_{k=1}^{M}\frac{1-a_k(\lambda)}{\lambda\, a_k(\lambda)}\right\}^{-1}.$$

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1. Spiegel, p. 20

From the steady-state probabilities (4.52) to (4.56) we really see that the effect of the type of repair and waiting time distributions on these probabilities has vanished. Only the mean values of repair and waiting times appear in the expressions of the probabilities. The failure rate of a S_1-component effects through the values of Laplace transform $a_k(\lambda)$.

4.5. On reliability of the system

Since both transient and steady-state probabilities of the system are known questions concerning the reliability of the system can be taken into consideration. One of the most important indices measuring the quantity 'reliability' in a quantitative sense is (pointwise) availability P_a of the system. Availability is defined[1] as the probability

$$(4.58)\quad P_a(t) = P\left\{\text{the system is operable at time } t\right\}.$$

From the definition of the states of the system it follows that the system is operable in being in one of the states 0,...,M-1 and inoperable in any other state. Availability of the system at any time t is so obtained as the probability that the system at that time is in one of the states 0,...,M-1.

On basis of equation (4.46) the Laplace transform of the transient state availability becomes

$$(4.59)\quad \begin{aligned} p_a(s) &= \sum_{m=0}^{M-1} p_m(s) = \sum_{m=0}^{M-1} \frac{\left[1 - a_{M-m}(s+\lambda)\right] Q(s,M-m-1)}{(s+\lambda)a_{M-m}(s+\lambda)\left[Q(s,M) - b(s)c(s)\right]} \\ &= \frac{1}{s+\lambda - \sum_{i=1}^{N} \lambda_i h_i(s) m_i(s)} \; \frac{Q(s,M) - 1}{Q(s,M) - b(s)c(s)} . \end{aligned}$$

For given probability densities characteristic of the system, availability $P_a(t)$ is after this obtained as the inverse transform of the right hand side of equation (4.59).

As far as the state probabilities $P_S(t)$ are already known (cf. equation (4.50)), availability is obtained through direct addition

1. Gnedenko et al, p. 110

(4.60) $$P_a(t) = \sum_{m=0}^{M-1} P_m(t) .$$

Instead of that the steady-state availability can be determined also in the case of general distributions in a closed form. From equation (4.52), we get

(4.61) $$P_a = \sum_{m=0}^{M-1} P_m = \frac{1}{1 + \sum_{i=1}^{N} \lambda_i(\overline{H}_i + \overline{M}_i) + S_M(\lambda)(\overline{B} + \overline{C})} .$$

Under the steady state, availability P_a has, along with the probability of operation at a certain moment, a sensible empirical interpretation. It gives the proportion of time that the system under the steady state is operable and in operation[1].

5. Conclusion

The paper has dealt with behaviour of a system with a fixed structure and with a spesific repair policy. The examinations have been based on the theory of stochastic processes and they have been carried out from the point of view of the system reliability. In formulation of the model an important part is played by the supplementary variable technique that has made it possible to give up using only the exponential distribution as a pattern for the system's random variables; examinations have been enlarged to cover all the continued distributions. The solutions of the model are based on the Laplace transform, especially at the steady state their special properties have been efficiently made use of. The examinations reveal also the methodologic nature of the model; for a system with other structure or other repair policy, it is easy to make a model of its own according to the general principles described above.

1. Gnedenko et al, p. 102

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