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OPTIMAL MAINTENANCE POLICY AND PLANNED SALE DATE FOR A MACHINE SUBJECT TO DETERIORATION AND RANDOM FAILURE

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1 Introduction

Simultaneous determination of an optimal maintenance policy and sale date of a machine has been the subject of considerable recent investigation, both because of the inherent practical importance of the question and because of the interesting mathematical problems posed. From the point of view of mathematics, these problems fall naturally within the framework of optimal control theory. This approach to simultaneous maintenance - sale date optimization was initiated by Näslund [4] and followed up by Thompson [6], Arora and Lele [2] and Kamien and Schwartz [3].

The model formulated in [4] is a very general one. It is assumed that the receipts obtained from the use of a machine are some function of the quality of the machine and the maintenance effort supplied. Further, the quality (salvage value) of the machine changes as a function of time and maintenance performed. Using Pontryagin's maximum principle, Näslund gives general guidelines for optimizing the maintenance policy and sale date of the machine; as a criterion of optimality he uses the present value of the machine, i.e. the sum of the discounted salvage value and the cumulative discounted net cash flow.

The main contribution in Thompson's model [6] lies in making the functional forms given by Näslund explicit, so that it is possible to obtain both qualitative (the bang-bang type maintenance policy) and quantitative (the time point maintenance is switched off, the sale date of the machine) solutions. The model suggested by Arora and Lele [2] is a modification of Thompson's model. The state equation describing the deterioration of the machine with time is made more general and more realistic at the same time. The type of the solution remains bang-bang although the number of qualitatively different solutions increases. Scott and Jefferson [5] have recently proposed a slightly modified bilinear model which provides a far richer spectrum of optimal maintenance policies than the earlier linear models in [6] and [2].

All the models above consider only the gradual deterioration of the machine, and the possibility of machine failure is not taken into account. Kamien and Schwartz [3] consider the failure part of the problem by taking the probability of machine failure as the state variable and by maximizing the present value of the expected returns from the machine. However, the model of Kamien and Schwartz does not take into account the possibility of decreasing the degradation of the machine by preventive maintenance. Also the rate of revenue of the machine is assumed to be independent of its age.

While Thompson, Arora and Lele, and Scott and Jefferson consider the gradual deterioration of the machine with time only, and Kamien and Schwartz the failure part only, Alam and Sarma [1] have tried to incorporate both features in a single model. They use the model of Arora and Lele and incorporate the machine failure probability into the performance index by taking the life time of the machine as a random variable. The most serious unsatisfactory aspect in this model is the fact that the sale date of the machine is not as an object of optimization: the machine is kept in use as long as it is operable, even though its use would not be profitable any more.

In the present paper we provide our own model, which extends and unifies the earlier results for the problem and represents a more realistic situation in practice. In section 2 we briefly review the earlier models [6], [2] and [1]. In section 3 we incorporate the concept of planned sale date and its optimization into the model and thus remove the possibility of unprofitable use of the machine which is included in the model [1]. In section 4 we derive the necessary conditions for the optimal maintenance policy and planned sale date. We also show that these conditions have interesting economic interpretations in terms of marginal costs and revenues. In section 5 we derive a detailed analytic solution for the problem in the special case, when the life time of the machine is exponentially distributed and independent of the maintenance. We also obtain that the solution of this stochastic model can be written in the same form as the solution of the corresponding deterministic model if we instead of the original discount rate use a "risk-adjusted" discount rate and instead of the real production rate use a "risk-adjusted" production rate. In section 6 we work out numerical examples to illustrate the results and to compare the different models. Finally, we summarize and comment on the results obtained.

2 Review of related models

Thompson considers the following problem: find the optimal maintenance policy $u(t)$ and the optimal sale date T for a machine to maximize the present value $V(T)$ of the machine given by

$$(1)\qquad V(T) = S(T)e^{-rT} + \int_0^T Q(t)e^{-rt}dt,$$

where the salvage value $S(t)$ and the net income function $Q(t)$ satisfy the relationships

$$(2)\qquad \frac{dS(t)}{dt} = -d(t) + f(t)u(t), \quad S(0) = S_0$$

and

(3) $Q(t) = pS(t) - u(t).$

In (1) to (3) r is the rate of interest (or discount rate), d(t) is the deterioration rate, f(t) is the maintenance effectiveness function and p is the (constant) production rate. The maintenance function u(t) (maintenance here means money spent over and above the minimum spent on necessary repairs) is the control variable satisfying for all t, $0 \le t \le T$, the requirement

(4) $0 \le u(t) \le U,$

and V(t) and S(t) are the state variables satisfying the differential equations (2) and

(5) $$\frac{dV(t)}{dt} = \{-d(t) + [f(t) - 1]u(t) + (p - r)S(t)\}e^{-rt}.$$

A straightforward application of the maximum principle gives the following optimal maintenance policy $u^*(t)$

(6) $$u^*(t) = \begin{cases} U, \text{ if } f(t) > g(T-t) \\ \text{arbitrary} \in [0, U], \text{ if } f(t) = g(T-t) \\ 0, \text{ if } f(t) < g(T-t), \end{cases}$$

where

(7) $$g(t) = \frac{r}{p - (p-r)e^{-rt}}.$$

With Thompson's assumptions: d, f and u are piecewise continuous, d is nondecreasing, and f is non-increasing, we get an optimal maintenance policy of one of the following three types:

(i) $u^*(t) = U$ for all $t \in [0, T]$

(ii) $u^*(t) = 0$ for all $t \in [0, T]$

(iii) $$u^*(t) = \begin{cases} U \text{ for } t \in [0, T') \\ \text{arbitrary for } t = T' \\ 0 \text{ for } t \in (T', T] \end{cases}$$

where T' satisfies the equation $f(T') = g(T-T')$.

The optimal sale date T is obtained as the solution of the equation

(8) $$S(T) = \begin{cases} \frac{d(T) - [f(T) - 1]U}{p - r} & \text{in the case (i)} \\ \frac{d(T)}{p - r} & \text{in the cases (ii) and (iii).} \end{cases}$$

We can generalize (as it is done by Arora and Lele) the model slightly by changing (2) to

(9) $$\frac{dS(t)}{dt} = -a(t) - bS(t) + f(t)u(t), \quad S(0) = S_o,$$

i.e. by considering the deterioration rate d(t) as made up of two components:

(10) $d(t) = a(t) + bS(t),$

where a(t) is the obsolescence rate which is due to technological process, and bS(t) is the depreciation rate which is due to change in physical characteristics and the performance of the machine. By an exponential transform (see Alam and Sarma [1]), equation (9) can be changed to

(11) $$\frac{dS_1(t)}{dt} = -a_1(t) + f_1(t)u(t),$$

where $S_1(t) = S(t)e^{bt}$, $a_1(t) = a(t)e^{bt}$ and $f_1(t) = f(t)e^{bt}$.

The performance index (1) can now be written in the form

(12) $$V(T) = S_1(T)e^{-(r+b)T} + \int_0^T [pS_1(t)e^{-(r+b)t} - u(t)e^{-rt}]dt.$$

The optimal control problem remains essentially the same, instead of the original functions we only use the transformed "1-indexed" functions and instead of the original discount rate r we use the effective discount rate r+b in connection with the transformed salvage value $S_1(t)$. However, there exists a qualitative difference between the solutions of the two models. This is due to the change in the nature of the maintenance effectiveness function. Whereas f(t) is normally a nonincreasing function of t, $f_1(t)$ can assume any arbitrary nature. The nature of the optimal maintenance policy remains bang-bang, but we may also obtain more than one switching point where the level of maintenance changes (see Arora and Lele [2]). The optimal maintenance policy can be derived from equations (6) and (7), if we in (7) instead of r use the effective discount rate r+b (cf. Alam and Sarma, eq. (7)). The optimal sale date T is obtained as the solution of the equation (cf. equation (8)):

(13) $$S(T) = \frac{a(T) - [f(T) - 1]u^*(T)}{p - r - b}.$$

In the model suggested by Alam and Sarma the variable T appearing in the performance index (12) is considered as a random variable, as the stochastic life time of the machine. This randomness is due to the chance of the machine failing. Thus the sale date of the machine is not considered or optimized, the machine is used until it fails and must be junked.

Let $p_T(t;u(\tau), 0 \le \tau \le t)$, $P_T(t;u(\tau), 0 \le \tau \le t)$ and $Q_T(t;u(\tau), 0 \le \tau \le t) = 1 - P_T(t;u(\tau), 0 \le \tau \le t)$ denote the density function, the cumulative distribution function and the complementary distribution (or reliability) function, respectively, of the random variable T. Further, let $p_T(t;u)$ and $Q_T(t;u)$ compactly represent the preceding quantities. The problem in the model [1] is to find the optimal maintenance policy u*(t) which maximizes the expectation of the present value given by

(14) $$E\{V(T)\} = \int_0^\infty p_T(t;u)\{S_1(t)e^{-(r+b)t} + \int_0^t [pS_1(s)e^{-(r+b)s} - u(s)e^{-rs}]ds\}dt$$

subject to (4) and (11). After some simplification, (14) can be written in the form

(15) $$E\{V(T)\} = \int_0^\infty \{[pQ_T(t;u) + p_T(t;u)]S_1(t)e^{-(r+b)t} - Q_T(t;u)u(t)e^{-rt}\}dt.$$

Explicit solutions for the problem are obtained in the case where $p_T(t;u)$ and, hence, $Q_T(t;u)$ are independent of u (see [1], pp. 173-174).

3 Model for the optimal maintenance policy and planned sale date of a machine subject to failure

The model of Alam and Sarma, which takes into account the probability of machine failure, does not, on the other hand, consider the possibility of selling the machine in such a case when it is still operable (i.e. not failed) but its use is highly unprofitable. In order to remove this unsatisfactory aspect from the model we introduce the concept of planned sale date. Thus, we seek a planned sale date T, and a maintenance policy u(t), $0 \le t \le T$, for the machine until it fails and is junked or is sold, whichever comes first. These are chosen to maximize the expected present value of the machine, the expectation being taken with respect to the random life time of the machine.

The state equation considered is again

(16) $$\frac{dS_1(t)}{dt} = -a_1(t) + f_1(t)u(t),\ 0 \le u(t) \le U,$$

where the quantities have the same meaning as in the previous models. Let T denote the planned sale date of the machine, i.e. T is the time at which the machine will be sold provided it is still working. Let τ denote the life time of the machine, τ is now a nonnegative random variable. If τ gets a value less than T, the machine is junked at the time of its failure. We assume, that the junk value of the machine at time t is equal to its salvage value S(t). The present value of the machine at a time t, when the machine is still operable, is (cf. equation (12)):

(17) $$V(t) = S_1(t)e^{-(r+b)t} + \int_0^t [pS_1(s)e^{-(r+b)s} - u(s)e^{-rs}]ds.$$

Let $V_0(T)$ be the present value really observed, when the planned sale date is T. Obviously we have

(18) $$V_0(T) = \begin{cases} V(T), & \text{if } \tau \ge T \\ V(\tau), & \text{if } \tau < T. \end{cases}$$

The performance index to be maximized, the expected present value of the machine, is now

(19) $$E\{V_0(T)\} = \int_0^T V(t)p_\tau(t;u)dt + \int_T^\infty V(T)p_\tau(t;u)dt = \int_0^T V(t)p_\tau(t;u)dt + Q_\tau(T;u)V(T).$$

Substituting (17) in (19) we get first

(20) $$E\{V_0(T)\} = \int_0^T p_\tau(t;u)S_1(t)e^{-(r+b)t}dt + Q_\tau(T;u)S_1(T)e^{-(r+b)T} + \int_0^t \{p_\tau(t;u)\int_0^t [pS_1(s)e^{-(r+b)s} - u(s)e^{-rs}]ds\}dt + Q_\tau(T;u)\int_0^T [pS_1(t)e^{-(r+b)t} - u(t)e^{-rt}]dt,$$

which after integration by parts in the second integral and after some simplifications becomes

(21) $$E\{V_0(T)\} = \int_0^T \{[pQ_\tau(t;u) + p_\tau(t;u)]S_1(t)e^{-(r+b)t} - Q_\tau(t;u)u(t)e^{-rt}\}dt + Q_\tau(T;u)S_1(T)e^{-(r+b)T}.$$

We can readily see that (21) is of the same form as (12) in the corresponding deterministic model only with the coefficients of $S_1(t)$ and u(t) modified. We also see that (21) gives (15), if we set $T = \infty$. Thus the model of Alam and Sarma is got as a special case of our model with the planned sale date fixed to infinity (whereas our aim is to optimize T also).

4 Analysis of the model

In the following we assume that the failure probability is not a function of maintenance. We can now obtain explicit solutions for the problem and perform a detailed analysis of the model which would not be possible in the general case. Therefore, let $p_\tau(t)$ and $Q_\tau(t)$ denote the density function and reliability function which don't depend on maintenance. Then (21) becomes

(22) $$E\{V_0(T)\} = \int_0^T [p_1(t)S_1(t) - Q_\tau(t)u(t)]e^{-rt}dt + Q_\tau(T)S_1(T)e^{-(r+b)T},$$

where we have denoted $p_1(t) = [pQ_\tau(t) + p_\tau(t)]e^{-bt}$. The Hamiltonian for this optimal control problem is

(23) $$H(S_1,\lambda,u,t) = -[p_1(t)S_1(t) - Q_\tau(t)u(t)]e^{-rt} + \lambda(t)[-a_1(t) + f_1(t)u(t)]$$

where the adjoint variable $\lambda(t)$ satisfies the differential equation

(24) $$\frac{d\lambda(t)}{dt} = -\frac{\partial H}{\partial S_1} = p_1(t)e^{-rt}$$

with terminal condition

(25) $$\lambda(T) = -\frac{\partial}{\partial S_1}[Q_\tau(T)S_1(T)e^{-(r+b)T}] = -Q_\tau(T)e^{-(r+b)T}.$$

Application of maximum principle (i.e. minimization of H with respect to u) gives the optimal maintenance policy u^*:

$$(26)\qquad u^*(t) = \begin{cases} U, \text{ if } Q_\tau(t)e^{-rt} + \lambda(t)f_1(t) < 0 \\ \text{arbitrary } \in [0,U], \text{ if } Q_\tau(t)e^{-rt} + \lambda(t)f_1(t) = 0 \\ 0, \text{ if } Q_\tau(t)e^{-rt} + \lambda(t)f_1(t) > 0 \end{cases}$$

Equation (26) shows that the optimal maintenance policy is still bang-bang with any finite number of switches from U to 0 and/or vice versa (as we will later in the case of exponentially distributed life time demonstrate). The adjoint variable $\lambda(t)$ in (26) is easily found from (24) and (25)

$$(27)\qquad \lambda(t) = -Q_\tau(T)e^{-(r+b)T} - \int_t^T [pQ_\tau(t) + p_\tau(t)]e^{-(r+b)t}dt.$$

Next we will consider the economic interpretation of the formula (26) (for similar economic evaluations of some replacement models see [7]). For notational simplificity, let us assume that there exists exactly one switching point T' $(0 < T' < T)$ at which the maintenance is changed from U to 0. At the moment T' we have, see equation (26),

$$(28)\qquad Q_\tau(T')Ue^{-rT'} + \lambda(T')f_1(T')U = 0.$$

Substituting (27) in (28) and rearranging terms, (28) can be written in the form

$$(29)\qquad Q_\tau(T')Ue^{-rT'} = \int_{T'}^T p_\tau(t)Uf(T')e^{-b(t-T')}e^{-rt}dt + Q_\tau(T)Uf(T')e^{-b(T-T')}e^{-rT} + \int_{T'}^T Q_\tau(t)pUf(T')e^{-b(t-T')}e^{-rt}dt.$$

One the left hand side of (29), U is the (last) maintenance expenditure spent at moment T', $Ue^{-rT'}$ is the present value of this expenditure, and $Q_\tau(T')$ is the probability that this expenditure will be spent (the machine is still working at T'). Writing

$$(30)\qquad Q_\tau(T')Ue^{-rT'} = Q_\tau(T')Ue^{-rT'} + [1-Q_\tau(T')]\cdot 0\cdot e^{-rT'} = E\{u^*(T')e^{-rT'}\},$$

we see that the left hand side of (29) can be interpreted as the expectation of the present value of the maintenance expenditure spent at the planned end of the optimal maintenance period. On the right hand side of (29), Uf(T') is the extra impulse in the salvage value of the machine yielding by the use of the maintenance expenditure U at moment T'. Further, $Uf(T')e^{-b(t-T')}$ is the value of this impulse still remaining at time $t > T'$ (the depreciation rate of the salvage value is b), and $Uf(T')e^{-b(t-T')}e^{-rt}$ is its present value. Using similar arguments as in deriving (30), we deduce that the right hand side of (29) can be interpreted as the expected value of the profit, which the use of the maintenance expediture U at moment T' yields on the present salvage value (the first two terms) and on the present value of the future operating receipts (the third term). The conclusion reached is thus: the planned ending point of the maintenance period (i.e. the moment maintenance is stopped for a still operable machine) has to be chosen

so that the expected present value of marginal maintenance outlay is equal to its effect on the expectation of the discounted salvage value plus its effect on the expected cumulative cash flow. Analogical interpretations can be found if there are another number or another type of switches in the optimal maintenance policy.

Equation (26) together with (27) gives the optimal maintenance policy with the planned sale date T considered as fixed. We still have to choose T so as to maximize $E\{V_0(T)\}$. To do this we differentiate $E\{V_0(T)\}$ in (22) with respect to T and set it equal to zero. Using similar reasoning as Thompson ([6], p. 546) we get first

$$(31)\qquad \frac{dE\{V_0(T)\}}{dt} = [p_1(T)S_1(T) - Q_\tau(T)u^*(T)]e^{-rT} + [\frac{dQ_\tau(T)}{dT}S_1(T) + Q_\tau(T)\frac{dS_1(T)}{dT} - (r+b)Q_\tau(T)S_1(T)]e^{-(r+b)T},$$

which after some labour becomes

$$(32)\qquad \frac{dE\{V_0(T)\}}{dT} = Q_\tau(T)\{[pS(T) - u^*(T)] + [f(T)u^*(T) - a(T) - bS(T)] - rS(T)\}e^{-rT}.$$

At the optimum we thus have

$$(33)\qquad \{pS(T) - u^*(T)\} + \{f(T)u^*(T) - [a(T) + bS(T)]\} = rS(T).$$

It is worth to note that the optimal planned sale date is independent of the distribution of the life time. Further, we can also obtain the following economic interpretation for this condition of optimality: the optimal planned sale date is reached at the moment when the net value of marginal operating receipts $(pS(T) - u^*(T))$ plus the net value of increase in the salvage value $(f(t)u^*(T) - [a(T) + bS(T)])$ are equal to the opportunity cost of the capital still invested at that moment $(rS(T))$. From (33) we can further see that the condition for the optimal planned sale date can also be written in form (13), i.e. in the form of the corresponding condition of the corresponding deterministic model (where T is the actual sale date, however).

5 A particular case: exponentially distributed life time

Equations (26) with (27) and (33) give us the general procedure to find out the optimal solution of the problem given in section 3. We shall now demonstrate the explicit calculation of the optimal maintenance policy and planned sale date for a particular probability density function, viz. for the exponential case.

Let the constant failure rate of the machine be σ, whereafter (for $t \geq 0$)

$$(34)\qquad p_\tau(t) = \sigma e^{-\sigma t}$$

and

$$(35)\qquad Q_\tau(t) = e^{\sigma t}.$$

By substituting (34) and (35) in (27) and integrating we get for the adjoint variable $\lambda(t)$ the expression

$$(36)\qquad \lambda(t) = -\frac{e^{-(r+b+\sigma)t}}{r+b+\sigma}\{p+\sigma-(p-r-b)e^{-(r+b+\sigma)(T-t)}\}.$$

Using (36) in (26), the following optimal maintenance policy $u^*(t)$ can be obtained

$$(37)\qquad u^*(t) = \begin{cases} U, \text{ if } f(t) > g(T-t) \\ \text{arbitrary } \in[0,U], \text{ if } f(t) = g(T-t) \\ 0, \text{ if } f(t) < g(T-t), \end{cases}$$

where

$$(38)\qquad g(t) = \frac{r+b+\sigma}{p+\sigma-(p-r-b)e^{-(r+b+\sigma)t}}.$$

It is worth to note that the optimal maintenance policy (37) is of the same form as that of the corresponding deterministic model presented by Arora and Lele (see p. 4 for comments on that optimal policy). The only difference between these two optimal policies is in the expression of the function $g(t)$: in our stochastic model we must instead of the effective discount rate $r+b$ use the risk-adjusted discount rate $r+b+\sigma$ and instead of the real production rate p use the risk-adjusted production rate $p+\sigma$. We thus get the following interesting economic interpretation for the failure rate parameter σ: by considering σ as a risk premium which must be added both to the discount rate and to the production rate, i.e. by substituting r by $r+\sigma$ and p by $p+\sigma$, we turn back to the deterministic case. Uncertainty in the life time of the machine reveals as a higher discount rate and a higher production rate in the model.

Next we shall consider the various forms which the optimal policy $u^*(t)$ may take. We assume, as before, that the maintenance effectiveness function $f(t)$ is a nonincreasing function of time. The nature of the function $g(t)$ depends on the values of the parameters p, r and b. We get three different cases depending on whether the production rate r is greater than or equal to or less than the effective discount rate $r+b$.

Case I: $p > r+b$. Now $g(t)$ is a monotonically decreasing function of time. Figure 1 shows the three qualitatively different possibilities for the optimal policy $u^*(t)$.

Case II: $p = r+b$. Now we have $g(T-t) \equiv 1$, $0 \le t \le T$. The optimal policy $u^*(t)$ has one of the three forms given in Fig. 1.

Case III: $p < r+b$. The function $g(t)$ is now monotonically increasing ($g(T-t)$ monotonically decreasing), otherwise it can be convex, concave or even s-shaped. This leads to various types of optimal maintenance policies. Some simple ones are presented in Fig. 2. One may also obtain more than two switching points where the level of maintenance changes.

The optimal planned sale date is obtained from (33), where $S(t)$ is the solution of the state equation (16) with $u(t)$ taken as the optimal policy $u^*(t)$. We can also write (33) in the form

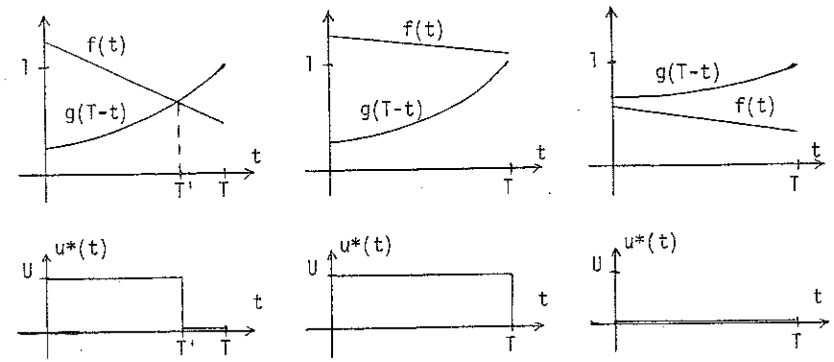

Figure 1. Optimal maintenance policies for $p > r+b$.

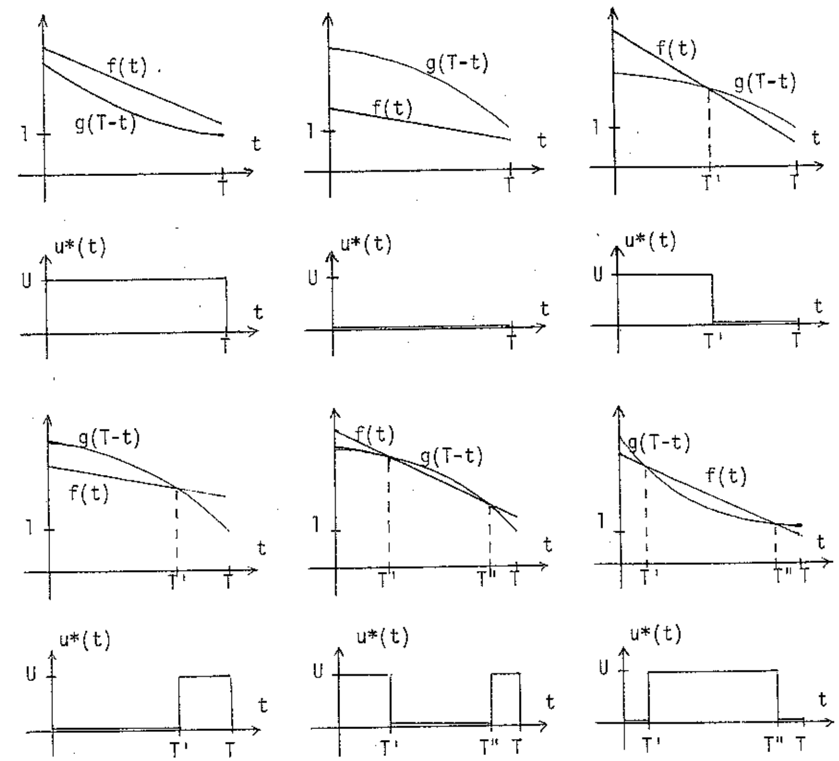

Figure 2. Some optimal maintenance policies for $p > r+b$.

(39) $\{(p+\sigma)S(T) - u^*(T)\} + \{f(T)u^*(T) - [a(T) + bS(T)]\} = (r+\sigma)S(T)$

to get an analogous risk-adjusted production rate and risk-adjusted discount rate interpretation as in connection with the optimal maintenance policy.

6 An example

We shall now work an example that illustrates both the similarities and differences between the models [6], [2], and [1] and our model. Suppose $p = 0.10$, $r = 0.05$, $a(t) \equiv 2$ (constant), $b = 0.03$, $\sigma = 0.04$, $S_0 = 100$, $U = 1$ and $f(t) = 1.5e^{-0.02t}$.

Thompson's model ($b = 0$, $\sigma = 0$). Using equations (1) - (8) we can show that there exists one switching point from $U = 1$ to 0 at $T' = 38.6$ in the optimal maintenance policy. The optimal sale date is found to be $T = 50.1$. We thus have

(40) $$u^*(t) = \begin{cases} 1, & \text{for } 0 \le t \le 38.6 \\ 0, & \text{for } 38.6 < t \le 50.1. \end{cases}$$

The salvage value of the machine is

(41) $$S(t) = \begin{cases} 175 - 2t - 75e^{-0.02t}, & \text{for } 0 \le t \le 38.6 \\ 140.3 - 2t, & \text{for } 38.6 < t \le 50.1 \end{cases}$$

and especially $S(50.1) = 40.1$. The maximum of the present value of the machine is $V(50.1) = 140.7$ so that the discounted profit from the use of the machine is $V(50.1) - S_0 = 40.7$.

The model of Arora and Lele ($\sigma = 0$). Applying the criteria of the present model we obtain that the optimal sale date is $T = 5.3$ and the optimal maintenance policy is $u^*(t) = U = 1$ throughout the period of operation. The shorter period of economic operation is now, of course, due to the positive depreciation rate $b = 0.03$. The salvage value of the machine is

(42) $$S(t) = 16.7e^{-0.03t} + 150e^{-0.02t} - 66.7, \quad 0 \le t \le 5.3,$$

and especially $S(5.3.) = 82.5$. The maximal present value of the machine is $V(5.3) = 110.5$. The discounted profit from the use of the machine is $V(5.3) - S_0 = 10.5$.

The model of Alam and Sarma. The life time of the machine is now an exponentially distributed ($\sigma = 0.04$) random variable. The machine is kept as long as it is operable, the selling of an still operable machine is not considered. We get (see equation (13) in [1]) the following optimal maintenance policy

(43) $$u^*(t) = \begin{cases} 1, & \text{for } 0 \le t \le 28.0 \\ 0, & \text{for } t > 28.0. \end{cases}$$

The salvage value (of an operable machine) proceeds as

(44) $$S(t) = \begin{cases} 16.7e^{-0.03t} + 150e^{-0.02t} - 66.7, & 0 \le t \le 28.0 \\ 215e^{-0.03t} - 66.7, & t > 28.0 \end{cases}$$

The expected present value of the machine, when the optimal maintenance policy (43) is followed, is got from (15) and is $E\{V(T)\} = 96.5$. We see that the expected profit $E\{V(T)\} - S_0 = -3.5$ becomes negative, the use of the machine is unprofitable! In the following we shall show that by taking also the (planned) sale date as an object of optimization (the model of section 3) the use of the machine can be made profitable.

The model with random failure and simultaneous optimization of maintenance and planned sale date. Now we consider our own model presented in section 3 to 5. Using the criteria derived for the case of an exponential life time we obtain $T = 5.3$ (T is independent of the life time distribution and numerically the same as in the model of Arora and Lele) and the optimal maintenance policy $u^*(t) = 1$, $0 \le t \le T = 5.3$. The salvage value of the machine proceeds, provided the machine has not failed, according to (42). The expected present value of the machine is $E\{V_0(T)\} = 101.1$, thus the expected profit becomes $E\{V_0(T)\} - S_0 = 1.1$. We see that optimization of the sale date also in the case of a stochastically failing machine is necessary: instead of the negative expected profit -3.5 (when the machine is kept as long as it is operable; the model of Alam and Sarma) we get a positive result 1.1 (when the machine is kept only until $T = 5.3$ or until it fails, whichever comes first). However, the expected profit ($= 1.1$) is less than the profit in the corresponding deterministic case ($= 10.5$) due to the possibility of a failure before the optimal sale date. As a conclusion, the example clearly points out the importance of the sale date optimization also in the case of random life time of the machine.

References

[1] Alam, M. and Sarma, V.V.S., "Optimum maintenance policy for an equipment subject to deterioration and random failure", IEEE Trans. Syst. Man Cybernet., Vol. SMC-4 (1974), No. 2, pp. 172-175.

[2] Arora, S.R. and Lele, P.T., "A note on optimal maintenance policy and sale date of a machine", Mgmt Sci., Vol. 17 (1970), No. 3, pp. 170-173.

[3] Kamien, M.I. and Schwartz, N.L., "Optimal maintenance and sale age for a machine subject to failure", Mgmt Sci., Vol. 17B (1971), No. 8, pp. 495-504.

[4] Näslund, B., "Simultaneous determination of optimal repair policy and service life", Swedish J. Econ., Vol. 68 (1966), No. 2, pp. 63-73.

[5] Scott, C.H. and Jefferson, T.R., "A bilinear control model for optimal maintenance", Int. J. Control, Vol. 30 (1979), No. 2, pp. 323-330.

[6] Thompson, G.L., "Optimal maintenance policy and sale date of a machine", Mgmt Sci., Vol. 14 (1968), No. 9, pp. 543-550.

[7] Verheyen, P.A., "Economic interpretation of models for the replacement of machines", Eur. J. Opl Res., Vol. 3 (1978), No. 6, pp. 150-156.