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Abstract

The paper deals with bounds on the probability values of a random variable in such a case, when information about the distribution of the variate is only scantily available. The best known result in this area is the celebrated Tchebycheff's inequality, which only assumes that the first and second moments (or the expectation and the variance) of the distribution must exist. The customary form of this inequality is

$$(1) \qquad P\{|X-\mu| \geq k\sigma\} \leq \frac{1}{k^2} ,$$

where $\mu = E(X)$ is the expectation and $\sigma = D(X)$ the standard deviation of the distribution, and k is an arbitrary positive number. Although the result of Tchebycheff's inequality cannot be improved in the general case, there exists a lot of stronger inequalities, when more information about the distribution is available. The most extensive results are those which depend on a knowledge of some higher moments or smoothness of the distribution.

The present work obtains one-sided generalizations of inequality (1) by assuming further information of the distribution, the information being as scarce as possible, however. The method is based on the geometry of the cumulative distribution function: the admissible cdf's, i.e. the cdf's fulfilling the assumptions set, can pass only through certain parts of the $(x,F(x))$-plane. The boundaries of the admissible cdf-region lead to certain extremal distributions, and hence to the required inequalities.
The new inequalities are more efficient than (1) in two senses, viz.

1^o we can obtain (non-trivial) one-sided bounds of the type (1) also for all values of k less than 1

2^o for the values of k between 1 and some k_1 $(k_1 > 1)$, we get stronger one-sided bounds than those resulting from (1).

The paper contains new one-sided bounds for the following general classes of distributions. All the distributions are, however, assumed to have a finite expectation and variance as is the case in inequality (1).

I. <u>All distributions with finite mean and variance</u>

$$(2) \qquad \left.\begin{matrix} P\{X-\mu \geq k\sigma\} \\ P\{X-\mu \leq -k\sigma\} \end{matrix}\right\} \leq \frac{2(\sqrt{1+k^2}-1)}{k^2} .$$

For the values of k between 0 and $\sqrt{5}/2$, the one-sided bounds (2) are stronger than the basic two-sided inequality (1).

II. Distributions symmetrical about mean

$$(3) \qquad P\{X-\mu \geq k\sigma\} = P\{X-\mu \leq -k\sigma\} \leq \begin{cases} 1/2, & \text{when } 0 \leq k \leq 1 \\ 1/2k^2, & \text{when } k>1 \end{cases}$$

Inequality (3) is a direct consequence from the symmetry of the distribution and the one-sided bounds are stronger than the two-sided bounds (1) for all values of k.

III. Distributions with semi-infinite range: $X \in [\mu-a\sigma, \infty)$ or $X \in (-\infty, \mu+a\sigma)$

For the case $X \in [\mu-a\sigma, \infty)$ we obtain, if $0 \leq a \leq 1$,

$$(4a) \qquad P\{X-\mu \geq k\sigma\} \leq \frac{a}{k+a},$$

and, if $a > 1$, we have

$$(4b) \qquad P\{X-\mu \geq k\sigma\} \leq \begin{cases} \frac{a}{k+a}, & \text{for } 0 \leq k \leq \frac{a}{a^2-1} \\ \frac{2\sqrt{a^2+ak(ak+1)}-2a-k}{ak^2}, & \text{for } k> \frac{a}{a^2-1} \end{cases}$$

Bounds (4a) and (4b) are stronger than the basic one-sided bounds resulting from (1) for the values of k between 0 and $(1+\sqrt{1+4a^2})/2a$ (inequality (4a)) or 0 and k_a (inequality (4b)), where $k_a > 1$ (the value of k_a depends on the value of a). The bounds for the case $X \in [-\infty, \mu+a\sigma)$ are analogical to the inequalities (4a) and (4b).