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ON RELIABILITY PROPERTIES OF A PARALLEL SYSTEM WITH STATES OF REDUCED EFFICIENCY

ILKKA VIRTANEN
Turku School of Economics, Turku, Finland

This paper deals with the operational behaviour and reliability of a system which has several different levels of performance, i.e. the system also possesses the property of operation with reduced efficiency. Using the supplementary variable technique and Laplace and discrete transforms, both the transient and steady state behaviour of the system are derived. On the basis of this behavioural information the reliability of the system is obtained. The reliability analysis of the system is carried out within the framework of generalized reliability concepts developed for operationally multi-stage systems.

Key words: Systems with reduced efficiency, Generalized reliability characteristics, Supplementary variables, Discrete transforms.

1. Introduction

One of the most important problem areas in reliability theory is the analysis of redundant systems, i.e. of systems which have several parallel components, see [2] — [8], [11] — [13]. This matter can be regarded as a direct consequence from the common occurrence of systems of this kind in practice: one method widely used for increasing the reliability of a system is to introduce redundancy among the strategic components of the system. The basic types of redundancy are redundancy in parallel and standby redundancy, see [9, p. 76] and [1, p. 162]. The so-called (k,n) — structure is also common, see [1, p. 216].

A system composed of parallel components is also the subject of this paper. The ordinary redundancy composition is not, however, now involved, but the operation of the system has had for technical reasons to be branched into parallel channels, where each branch is responsible for a given portion of the capacity of the system. The failure of a branch (component) results now immediately in a reduction in the efficiency of the system. The reduction in efficiency is, however, only partial, amounting to the capacity of the failed component. Only the simultaneous failure of all the components makes the system totally inoperable. The present system has been first introduced and it has been in more detail described in a recent report by the author, see subsystem S_4 in [14, pp. 8—9 and 50—52]. Separation of the subsystem out of its complex entirety and consideration of the subsystem as an independent system make it now possible to bring the analysis to an end in a closed form.

Thus we have a system which has several possible levels of performance, ranging from full (capacity level) operability to total inoperability. The ordinary characteristics of system reliability are found to be insufficient to cover such a system: the reliability of the system remains unresolved or it gets a value that contradicts empirical observation and logical reasoning. Therefore, the reliability analysis of the system is carried out within the framework of the extended and generalized concepts of reliability which have been developed for systems with several possible levels of performance, see [14, pp. 30—48].

2. Mathematical model for the behaviour of the system

2.1. Description of the problem and assumptions in the model

The problem to be dealt with is as follows. The system consists of N ($N \geqslant 1$) identical independent components which are connected in parallel so that each one of them covers one Nth part of the capacity of the system. Thus the failure of one component reduces the level of performance of the system by an amount $1/N$ of the full capacity. When all N components have failed, the system becomes totally inoperable.

The failure times of each individual component are identically and independently distributed random variables having a common general distribution, i.e. the type of the distribution has not been specified. The repair times of the system are distributed according to another independent general distribution.

The system is assumed to be maintained by a single repair facility. Further it is assumed that the repair of any component is possible only when the whole system stops operating and that the repair of the system does not start until all N components have failed. After completion of the repair of all N components the system is put into operation again.

The object of the study is to find out both the transient and steady state behaviour of the system, and, on the basis of this information, to analyze the reliability properties of the system. This analysis is carried out using the generalized characteristic 'mean availability of the capacity' (see [14, p. 37]) as the criterion for the system reliability.

2.2. Notation

Let the possible levels of performance of the system be denoted by $c_0, c_1, \ldots, c_N$. Then we have

$$c_n = (1-n/N)C, \; n = 0,1,\ldots,N, \tag{1}$$

where C is the capacity of the system and n is the number of failed components attaining at the level c_n of performance. The corresponding proportional levels of performance of the system are denoted by $w_0, w_1, \ldots, w_N$, when

$$w_n = 1 - n/N, \; n = 0,1,\dots,N. \tag{2}$$

The behaviour of the system is described by state probabilities, the state (at time t) giving the number of failed components at that time. From (1) we see, that for each level of performance there is one and only one possible state. Let us define

$P_n(t)$ = probability that at time t n of the N components have failed; the system is fully operable ($n=0$) or operates at a reduced level of performance ($n=1,2,\dots,N-1$),

$p_n(x,t)dx$ = joint probability that at time t the system operates with n components having failed and the elapsed time since the system was last put into operation lies between x and $x+dx$ ($n=0,1,\dots,N-1$),

$P_N(t)$ = probability that at time t all the components have failed; the system is inoperable and under repair,

$p_N(x,t)dx$ = joint probability that at time t the system is inoperable and under repair, and the elapsed repair time lies between x and $x+dx$.

Let further be

$\alpha(x)$ = failure rate of a single component,

$a(x)$ = probability density of the failure time distribution of a single component,

$\beta(x)$ = repair rate of the system, i.e. of all N components together,

$b(x)$ = probability density of the repair time distribution.

2.3. The model

The model describing the behaviour of the system can be shown to be mathematically analogous to the model dealing with a parallel redundant system with N identical independent components (for derivation of that model see [13]). The analogy holds, however, only for the derivation of the state probabilities; the reliability characteristics of the system, the properties of the states etc. are, of course, quite different. In what follows, the different stages in the modelling are considered only briefly, the details can be found in the analogical "redundancy in parallel"-model.

The model has a form of the following set of differential-difference equations with variable coefficients (for any values of $x \geq 0$, $t \geq 0$)

$$[\partial/\partial x + \partial/\partial t + (N-n)\alpha(x)]p_n(x,t) = (1-\delta_{n,0})\alpha(x)(N-n+1)p_{n-1}(x,t),$$
$$n = 0,1,\dots,N-1, \tag{3}$$

$$[\partial/\partial x + \partial/\partial t + \beta(x)]p_N(x,t) = 0. \tag{4}$$

Consideration of the repair completion and the occurrence of the failure of a still operable component leads to the boundary conditions (for $x=0$, $t>0$)

$$p_0(0,t) = \int_0^{\infty} p_N(x,t)\beta(x)dx, \tag{5}$$

$$p_n(0,t) = \delta_{n,N} \int_0^{\infty} p_{N-1}(x,t)\ \alpha(x)dx,\ n = 1,2,\ldots,N. \tag{6}$$

When we assume that the system is without failure at the outset, we get the initial conditions (for all $x \geq 0$)

$$p_n(x,0) = \delta_{n,0}\,\delta(x),\ n = 0,1,\ldots,N. \tag{7}$$

In equations (3), (6) and (7) $\delta_{i,j}$ and $\delta(x)$ are the Kronecker delta and the Dirac delta function, respectively.

Applying the Laplace transform (the Laplace transform of a function $f(t)$ is denoted by $\bar{f}(s)$) and the discrete transforms

$$\overline{q_k}(x,s) = \sum_{n=0}^{N-k} \binom{N-n}{k} \overline{p_n}(x,s), \quad k = 1,2,\ldots,N \tag{8}$$

to equation (3), we first obtain for $\overline{q_k}(x,s)$, $k = 1,2,\ldots,N$:

$$\overline{q_k}(x,s) = \begin{cases} \overline{q_k}(0,s), & x = 0 \\ \left[\overline{q_k}(0,s) + \binom{N}{k} \right] \exp\left\{ -sx - \int_0^{x} k\alpha(x)\,dx \right\}, & x > 0, \end{cases} \tag{9}$$

where the $\overline{q_k}(0,s)$'s are not known yet. Applying the Laplace transform to equations (4) — (6), using (9) and the inverse discrete transforms of (8), viz.

$$\overline{p_n}(x,s) = \sum_{k=N-n}^{N} (-1)^{k-N+n} \binom{k}{N-n} \overline{q_k}(x,s), \quad n = 0,1,\ldots,N-1, \tag{10}$$

we get for $\overline{q_k}(0,s)$

$$\bar{q}_k(0,s) = \binom{N}{k} [D(s)]^{-1}\, \bar{b}(s) \sum_{j=1}^{N} (-1)^{j-1} \binom{N}{j} \overline{a_j}(s),\ k = 1,2,\ldots,N, \tag{11}$$

where $\bar{b}(s)$ is the Laplace transform of the repair time density function $b(x)$, $\bar{a}_j(s)$ is the Laplace transform of the failure density function $a_j(x)$,

$$a_j(x) = j\alpha(x) \exp\left\{ -\int_0^{x} j\alpha(x)dx \right\},\quad j = 1,2,\ldots,N, \tag{12}$$

and

$$D(s) = 1 - \bar{b}(s) \sum_{j=1}^{N} (-1)^{j-1} \binom{N}{j} \bar{a}_j(s). \tag{13}$$

Using (11) in (9), (9) in (10) and simplifying, we at last obtain the Laplace transforms of the state probabilities

$$\overline{P_n}(s) = \binom{N}{n} [sD(s)]^{-1} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} [1 - \bar{a}_{N-k}(s)],\ n=0,1,\ldots,N-1, \tag{14}$$

$$\overline{P_N}(s) = [1-\bar{b}(s)][sD(s)]^{-1} \sum_{k=1}^{N} (-1)^{k-1} \binom{N}{k} \bar{a}_k(s). \tag{15}$$

Equations (14) and (15) give the solution of the model in the Laplace transform domain. With given values of $a(x)$ and $b(x)$, (14) and (15) can be inverted to give the solution in the time domain, i.e. the state probabilities of the system.

3. Reliability analysis of the system

3.1. The reliability characteristic mean availability of the capacity

The system under study is a system which also possesses the property of operation with reduced efficiency, even at several different levels of performance. Because of this we introduce for the reliability analysis of the system the extended concepts of reliability developed for such systems with many different levels of performance (for derivation and use of these extended concepts see [14, sections 32 and 43]).

As the specific quantitative characteristic of reliability we use a generalized availability characteristic, the mean availability of the capacity $A_C(t)$, which is defined to be (see [14, p. 39]) the expected value of the proportional level of performance of the system at time t

$$A_C(t) = \sum_{n=0}^{N} w_n P_n(t). \tag{16}$$

$A_C(t)$ can thus be derived directly from the structural and behavioural properties (from the proportional levels of performance and the state probabilities) of the system.

3.2. Transient state reliability

In a general case, without any further knowledge about the failure and repair time distributions, the transient-state state probabilities can be obtained only in the Laplace transform domain. Substituting (2) and (14) — (15) in the Laplace transform of (16), we get the Laplace transform of the transient state mean availability of the capacity

$$\bar{A}_C(s) = \sum_{n=0}^{N} w_n \bar{P}_n(s)$$

$$= \sum_{n=0}^{N-1} (1-n/N) \binom{N}{n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} [sD(s)]^{-1}[1-\bar{a}_{N-k}(s)]$$

$$= \sum_{n=0}^{N-1} \binom{N-1}{n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} [sD(s)]^{-1}[1-\bar{a}_{N-k}(s)]. \qquad (17)$$

Again, with given failure and repair densities $a(x)$ and $b(x)$, equation (17) may be inverted to give the mean availability of the capacity function $A_C(t)$.

3.3. Steady state reliability

The steady state behaviour of the system can be found out using the final value theorem of Laplace transforms (see [10, p. 20]):

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\bar{f}(s). \qquad (18)$$

Applying this relation to the state probabilities (14), we get

$$\bar{P}_n = \lim_{t\to\infty} P_n(t) = \lim_{s\to 0} s\bar{P}_n(s)$$

$$= \binom{N}{n} \bar{D}^{-1} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \bar{A}_{N-k}, \; n = 0,1,\ldots,N-1, \qquad (19)$$

where

$$\bar{D} = \lim_{s\to 0} \left\{ D(s)/s \right\} = \bar{B} + \sum_{k=1}^{N} (-1)^{k-1} \binom{N}{k} \bar{A}_k, \qquad (20)$$

$$\bar{B} = \lim_{s\to 0} \left\{ [1-\bar{b}(s)]/s \right\} = \int_0^\infty x b(x) dx \qquad (21)$$

is the mean repair time of the system and

$$\bar{A}_k = \lim_{s\to 0} \left\{ [1-\bar{a}_k(s)]/s \right\} = \int_0^\infty x a_k(x) dx, \; k = 1,2,\ldots,N \qquad (22)$$

is the mean failure time (for the first component failure) of the system with k still operable components. In order to (19) be valid, the limiting values in (20)—(22) must, of course, exist.

The steady state value for the mean availability of the capacity thus becomes

$$\bar{A}_C = \bar{D}^{-1} \sum_{n=0}^{N-1} \binom{N-1}{n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \bar{A}_{N-k}. \tag{23}$$

From (23) we see that only the mean values (21) and (22) are needed for calculating $\bar{A}_C$; both the types and exact forms of the distributions may be unknown.

3.4. *Particular case*

If both the failure and repair times follow exponential distributions with parameters α and β, respectively, we have

$$\bar{A}_k = 1/k\alpha, \; k = 1,2,\ldots,N, \tag{24}$$

$$\bar{B} = 1/\beta. \tag{25}$$

Substituting (24) and (25) in (23), we get for the exponential system the steady state value of the mean availability of the capacity

$$\bar{A}_C = \frac{\sum_{n=0}^{N-1} \binom{N-1}{n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} [(N-k)\alpha]^{-1}}{(1/\beta) + \sum_{k=1}^{N} (-1)^{k-1} \binom{N}{k} (1/k\alpha)} = \left[(\alpha/\beta) + \sum_{k=1}^{N} (1/k) \right]^{-1} \tag{26}$$

From (26) we can see that in the exponential case the steady state value of A_C depends only on the number of parallel components and on the ratio of the failure and repair rates. Further we see that, with a fixed value for the ratio of α and β, $\bar{A}_C$ is a decreasing function of N, the number of parallel components in the system.

4. Discussion

Above we have considered a system which consisted of several independent parallel components, each one of which covered equal portions of the capacity of the system. Thus the composition was not a redundancy composition; we had had for operational and technical reasons to use more than one component.

As a result of the reliability analysis we saw, explicitly in the exponential case, that the composition in question lead to a reduction in the reliability of the system: the more components we had in the system, the lower was the reliability achieved. The decrease in reliability came from the fact that, with a great number of components, the system was found to operate for the majority of time with reduced efficiency.

The reliability of the system may be increased, however, if it is possible to repair the system during its operation: whenever any of the components fails, it is taken up for repair at once and repaired in turn by the repair facility; as soon as the component has been repaired it is reentered in the system. Some derivations concerning the latter system has already been made and the results will be published in a near future.

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