On relationships between ROI and IRR under inflation: A constant real cash-flow case

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The paper deals with two indices commonly used for measuring the profitability of an investment: the return on investment (ROI) and the internal rate of return (IRR). ROI may be considered as a simplified approximation for IRR and is commonly used instead of that. There are, however, several factors which cause differences between these two measures of profitability:
- the time dimension of cash-flows: the IRR-method takes the time value of money into account whereas in ROI no discounting occurs;
- the depreciation method: IRR is based on annuity depreciation, in ROI the straight line depreciation is usually applied;
- the service life of the investment: due to the lack of discounting in ROI, the two indices differ more the longer the service life;
- inflation: under inflation ROI also contains apparent profitability due to nominal increase of the cash-flows.

In the paper a mathematical model is constructed for comparing the behaviour of ROI and IRR in the constant real cash-flow case. In the model the difference between ROI and IRR is expressed as a function of three parameters: the profitability of the investment, the service life and the rate of inflation. Both analytical and simulated numerical solutions for the model are derived. On the basis of the results a clear description of the effects of the parameters on the relationships between ROI and IRR is obtained and, thus, several conclusions and recommendations for the usefulness of ROI can be made.

1. Introduction

The methods of internal rate of return (IRR) and return on investment (ROI) are both widely used in capital budgeting. ROI, a simplified form of IRR, is commonly expressed as a ratio between the annual return on investment (net of depreciation but including interest payments) and either the initial capital expenditure or the average capital invested in the asset. The internal rate of return, in turn, is defined as that rate of discount which equates the present value of the net cash inflows from the investment with its initial capital expenditure.

Traditionally, these methods have been compared as profitability indicators for the entire firm or department (Solomon (1975, pp. 236–248); Gordon (1974, pp. 343–356)), which implies that the investment concept to be used in the calculations consists of all the projects undertaken by the relevant unit. If an individual investment project is analysed, the profitability calculations are somewhat simplified, since the problems of treating growth may be ignored. In this case, the ROI and IRR measures of profitability may differ due to the following reasons:

(1) The time value of money. In ROI, the annual cash-flows are given equal weights irrespective of the time of their actualisation, whereas the IRR method applies a geometrically declining weight to the annual items. This weight is determined by the internal rate of return.

(2) The method of depreciation. IRR employs the annuity method of depreciation (sinking fund), while ROI is usually based on straight line depreciation.

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(3) The service life of the investment. The longer the life of the investment, the less weight are the cash inflows actualising during the final years of the economic life given in the IRR method. ROI, on the other hand, allocating the same weight to all cash-flow items, is less influenced by the length of the service life.

(4) Inflation increases the nominal net return on investment and thus also the nominal ROI becomes higher (Carsberg and Hope (1976, p. 34)). While also IRR is increased, the inflationary effects are not identical in these two cases, since inflation affects the time distribution of the nominal cash-flows.

This study examines the relationships between ROI and the IRR as profitability indicators in the case where the profitability of a single investment project is of interest. In order to make an analytical study possible some assumptions about the annual cash inflows must be made. The real cash inflows are assumed to be constant (which does not seem unduly unrealistic), and the nominal cash inflows grow at the rate of inflation. This study concentrates on specifying and analyzing the relationships between ROI and IRR. The general properties and problematics of IRR are not examined in this study (see Clark, Hindelang and Pritchard (1979, pp. 104–109)).

The following analyses are based on a pre-tax situation. Consequently, all the indicators to be discussed express the profitability of the investment before taxes.

2. The relationships between ROI and IRR: Derivation and analysis of the model

2.1. Derivation

It is assumed that the investment project under examination creates revenue over a period of n years, after which its salvage value equals zero. The annual cash inflows in real terms are assumed constant ($= P_0$ in the money of year 0). The nominal cash inflow grows at the rate of inflation ($= s$). Inflation is considered as a continuous quantity, since it is essentially continuous by character. Therefore, the nominal cash inflow in year t may be expressed as

$$P_t = P_0 \, \mathrm{e}^{st}, \tag{2.1}$$

where e is the base of the system of natural logarithms.

IRR may be determined by solving for that rate of discount, r, which makes the present value of the sum of annual nominal cash inflows just equal to the initial net cash outlay of the investment. Thus, the present value of P_t, the nominal cash inflow in year t, can be written as (continuous discounting is also here utilized)

$$\mathrm{NPV}(P_t) = P_t \, \mathrm{e}^{-rt}, \tag{2.2}$$

and, after taking (2.1) into account, as

$$\mathrm{NPV}(P_t) = P_0 \, \mathrm{e}^{-(r-s)t}, \tag{2.3}$$

where r denotes the nominal internal rate of return. The real internal rate of return is given by $r - s$. The present value of the sum of all annual cash inflows is now

$$\mathrm{NPV}(P) = \sum_{t=1}^{n} \mathrm{NPV}(P_t) = P_0 \sum_{t=1}^{n} \mathrm{e}^{(s-r)t} = \begin{cases} P_0 \, \mathrm{e}^{s-r} \dfrac{1 - \mathrm{e}^{(s-r)n}}{1 - \mathrm{e}^{s-r}}, & s \neq r, \\ nP_0, & s = r. \end{cases} \tag{2.4}$$

The derivation of equation (2.4) as well as the entire analysis in the paper is based on discrete flows and continuous discounting. The usage of continuous discounting can be justified by its mathematical advantages, see for example Beenhakker (1976, pp. 23–26). The equivalent discrete formulae may be derived by simple interest transformations.

In determining ROI, the sum of the nominal annual cash inflows is required instead of its present value.

This sum can be written as

$$\mathrm{SUM}(P)=\sum_{t=1}^{n} P_t = P_0 \sum_{t=1}^{n} \mathrm{e}^{st} = \begin{cases} P_0\, \mathrm{e}^{s} \dfrac{1-\mathrm{e}^{sn}}{1-\mathrm{e}^{s}}, & s \neq 0, \\ nP_0, & s=0. \end{cases} \tag{2.5}$$

For calculating ROI, the arithmetic mean of SUM(P) is required:

$$\bar{P}=\mathrm{SUM}(P)/n. \tag{2.6}$$

In order to arrive at the concept to be used in the numerator of ROI, the annual depreciation, expressed as straight line depreciation, must be deduced from the mean cash inflow. The annual straight line depreciation may be expressed as

$$D_t=\bar{D}=\mathrm{NPV}(P)/n, \tag{2.7}$$

where the initial capital expenditure has been replaced by the present value expression (2.4) since these two are equal due to the definition of IRR.

If ROI is defined using the concept of initial capital expenditure in the denominator (for other ways of defining ROI, see Clark, Hindelang and Pritchard (1979, pp. 52–53)), it equals

$$\mathrm{ROI}=\frac{\bar{P}-\bar{D}}{\mathrm{NPV}(P)}=\frac{\mathrm{SUM}(P)-\mathrm{NPV}(P)}{n\cdot \mathrm{NPV}(P)}. \tag{2.8}$$

By substituting the previously derived expressions for SUM(P) and NPV(P) into (2.8), ROI becomes

$$\begin{aligned}\mathrm{ROI}&=\frac{P_0\,\mathrm{e}^{s}\dfrac{1-\mathrm{e}^{sn}}{1-\mathrm{e}^{s}}-P_0\,\mathrm{e}^{s-r}\dfrac{1-\mathrm{e}^{(s-r)n}}{1-\mathrm{e}^{s-r}}}{n\cdot P_0\,\mathrm{e}^{s-r}\dfrac{1-\mathrm{e}^{(s-r)n}}{1-\mathrm{e}^{s-r}}}\\ &=\frac{\mathrm{e}^{rn}(1-\mathrm{e}^{sn})(\mathrm{e}^{r}-\mathrm{e}^{s})-(1-\mathrm{e}^{s})(\mathrm{e}^{rn}-\mathrm{e}^{sn})}{n(1-\mathrm{e}^{s})(\mathrm{e}^{rn}-\mathrm{e}^{sn})}.\end{aligned} \tag{2.9}$$

Further, denoting the discount factor e^{r} by R and the inflation factor e^{s} by S, one arrives at

$$\mathrm{ROI}=\frac{R^n(1-S^n)(R-S)-(1-S)(R^n-S^n)}{n(1-S)(R^n-S^n)}=\frac{1}{n}\left[\frac{R^n(1-S^n)(R-S)}{(1-S)(R^n-S^n)}-1\right]. \tag{2.10}$$

The above expression for ROI is applicable as long as $s \neq 0$ (we consider only the case $s \geqslant 0$) and $r \neq s$. If $s=0$ or $r=s$, (2.10) becomes $0/0$, and it must be calculated using the values $\mathrm{SUM}(P)=nP_0$ or $\mathrm{NPV}(P)=nP_0$, respectively, or by evaluating the expression as the limits when $s \to 0$ or $r \to s$, in the manner to be presented later. These exceptions do not represent, however, any anomalies (e.g. discontinuities) in the behaviour of ROI.

2.2. ROI as a function of IRR, the case of stable prices

When there is no inflation, i.e. $s=0$ ($S=1$), the nominal annual cash inflow remains constant and equals the real cash inflow, i.e. $P_t=P_0$. Expression (2.10) may be used to obtain ROI as

$$\mathrm{ROI}_{s=0}=\frac{1}{n}\left[\frac{R^n n(R-1)}{R^n-1}-1\right]=\frac{R^n(R-1)}{R^n-1}-\frac{1}{n}=\frac{1}{\bar{a}_{\overline{n}|r}}-\frac{1}{n}, \tag{2.11}$$

where $\bar{a}_{\overline{n}|r}=(R^n-1)/(R^n(R-1))$ is the present value factor for discrete payments under continuous discounting.

In order to compare ROI and the internal rate of return, r, the difference ROI-r is considered. The difference is considered as a function of r only because the main interest is in the relationship between ROI and r. The other influencing factor n is considered as a parameter. We obtain

$$g(r)=\text{ROI}(r)-r=\frac{e^{rn}(e^r-1)}{e^{rn}-1}-\frac{1}{n}-r. \tag{2.12}$$

If $g(r)=0$, both measures of profitability give identical readings. If $g(r)>0$, ROI indicates higher profitability, while if $g(r)<0$, IRR yields higher profitability. Using again R to denote e^r, one can write

$$G(R)=\text{ROI}(R)-\ln R=\frac{R^n(R-1)}{R^n-1}-\frac{1}{n}-\ln R. \tag{2.13}$$

Next, the function $g(r)$ is evaluated at the boundaries of the range $0 \leqslant r < \infty$:

$$g(0)=G(1)=\frac{1}{n}-\frac{1}{n}-0=0, \tag{2.14}$$

$$\begin{aligned}\lim_{r\to\infty} g(r)=\lim_{R\to\infty} G(R)&=\lim_{R\to\infty}\left[\frac{R^{n+1}-R^n}{R^n-1}-\frac{1}{n}-\ln R\right]\\ &=\lim_{R\to\infty}\left[\frac{R-1}{1-R^{-n}}-\frac{1}{n}-\ln R\right]=\infty.\end{aligned} \tag{2.15}$$

Thus, the difference between ROI and the internal rate of return has the end-point values 0 and ∞. The function $G(R)$ is next analysed within the interval $[1, \infty)$. The derivative of G is

$$\begin{aligned}\frac{dG}{dR}&=\frac{[(n+1)R^n-nR^{n-1}][R^n-1]-nR^{n-1}[R^{n+1}-R^n]}{(R^n-1)^2}-\frac{1}{R}\\ &=\frac{R^{2n+1}-R^{2n}-(n+1)R^{n+1}+(n+2)R^n-1}{R^{2n+1}-2R^{n+1}+R}.\end{aligned} \tag{2.16}$$

This is undetermined (of the form 0/0) when R equals one. By applying l'Hôpital's rule, one gets first

$$\left[\frac{dG}{dR}\right]_{R=1}=\left[\frac{(2n+1)R^{2n}-2nR^{2n-1}-(n+1)^2R^n+n(n+2)R^{n-1}}{(2n+1)R^{2n}-2(n+1)R^n+1}\right]_{R=1} \tag{2.17}$$

which still remains in the form 0/0. By further differentiation, one obtains

$$\begin{aligned}\left[\frac{dG}{dR}\right]_{R=1}&=\left[\frac{(2n+1)2nR^{2n-1}-2n(2n-1)R^{2n-2}-(n+1)^2nR^{n-1}+(n+2)n(n-1)R^{n-2}}{(2n+1)2nR^{2n-1}-2(n+1)nR^{n-1}}\right]_{R=1}\\ &=-\frac{n(n-1)}{2n^2}<0.\end{aligned} \tag{2.18}$$

It may be seen that the derivative dG/dR is negative at $R=1$ except for the case where $n=1$, when it takes the value zero. For all values of n, $n>1$, we thus have

$$\left[\frac{dg}{dr}\right]_{r=0}=\left[\frac{dG}{dR}\right]_{R=1}<0. \tag{2.19}$$

As r grows from the value zero (the internal rate of return on the investment becomes positive), the difference $g(r)=\text{ROI}-r$ becomes negative. Thus, if the investment has a low positive profitability, the internal rate of return is higher than ROI. By using the values $g(r)=0$, $[dg/dr]_{r=0}<0$, $\lim_{r\to\infty} g(r)=\infty$, and the general expression for $g(r)$, i.e.

$$g(r)=\frac{e^r-1}{1-e^{-rn}}-\frac{1}{n}-r, \tag{2.20}$$

it is possible to describe the general behaviour of $g(r)$ as shown in Fig. 1. The function $g(r)$ emanates from the origin, is negative over some range $0 < r < r_0$ (where ROI $< r$), equals zero at the point $r = r_0$, and becomes positive (ROI $> r$) as $r > r_0$. The largest negative value of the function in absolute terms, g_m, is attained at a point $r = r_m$.

Next, the values r_0 and r_m are determined. The value r_0 is the (positive) root of the equation

$$g(r) = \frac{e^r - 1}{1 - e^{-rn}} - \frac{1}{n} - r = 0 \tag{2.21}$$

which cannot be solved analytically. However, a numerical solution is possible for a given value of n. For example, if $n = 10$, r_0 equals 0.398. Thus, if the length of life for the investment under examination is ten years, and its internal rate of return is 39.8%, its ROI measures profitability in exactly the same way as the internal rate of return. Likewise, the internal rate of return for this investment exceeds its ROI in the range $0 < 100r < 39.8\%$. Thus, for small and moderate values of r, ROI underestimates the internal rate of return; compare with some earlier results based on discrete mathematics (e.g. Merrett and Sykes (1960, p. 104) and Sarnat and Levy (1969, pp. 483–484)). The intensive growth of $g(r)$ for high values of r is, however, typical for the expression (2.12) adopted.

The largest negative value of the function $g(r)$ in absolute value, g_m, corresponds to the value of the internal rate of return which is determined by the condition

$$\left[\frac{dg}{dr}\right]_{r=r_m} = 0 \tag{2.22}$$

or (when R_m is used to denote e^{r_m})

$$\left[\frac{dG}{dR}\right]_{R=R_m} = 0. \tag{2.23}$$

The solution for R_m is derived from

$$R^{2n+1} - R^{2n} - (n+1)R^{n+1} + (n+2)R^n - 1 = 0, \tag{2.24}$$

which, once again, must be solved numerically. Once R_m is solved for, r_m may be determined from $r_m = \ln R_m$. In the case where $n = 10$, R_m equals 1.215 and thus $r_m = 0.195$. The difference between the profitability indicators ROI and r achieves its largest negative value at $100r = 19.5\%$ and equals $g(0.195) = \text{ROI}(0.195) - 0.195 = 0.151 - 0.195 = -0.044$. In other words, the internal rate of return equals 19.5% and ROI 15.1%.

Let us now consider the effects of changing the service life, n, on the values of r_0, r_m and g_m. This analysis has some common features with Figure 1 in Merrett and Sykes (1960), where the effect of the service life on the relationships between ROI and IRR is schematically sketched out and with Table 2 in Sarnat and Levy (1969), which gives deviations of ROI from IRR for selected values of IRR and n. Table 1 presents the values of r_0, r_m, ROI(r_m) and g_m for specific values of n. The function $g(r)$ behaves exceptionally in the case where the service life has the length of one year (cf. with the previous discussion

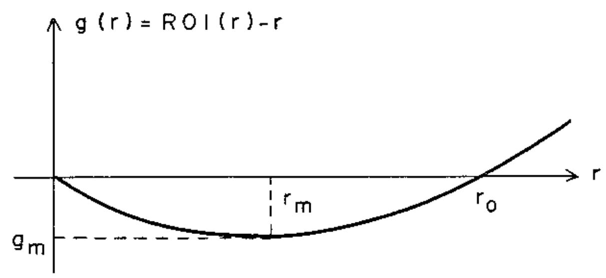

Fig. 1. The difference $g(r)$ as a function of the internal rate of return, r.

Table 1
The values of r_0, r_m and g_m as functions of the service life, n

n	$100r_0$ (%)	$100r_m$ (%)	$100\text{ROI}(r_m)$ (%)	$\lvert 100g_m \rvert$ (%)
1	–	–	–	–
2	42.5	22.1	19.2	2.9
3	49.6	25.8	21.3	4.5
4	49.9	25.9	20.8	5.1
5	48.5	25.0	19.8	5.2
10	39.8	19.5	15.1	4.4
20	29.8	13.4	10.4	3.0
40	21.5	8.5	6.7	1.8
100	13.8	4.4	3.6	0.8
∞	0	0	0	0

on $\mathrm{d}g/\mathrm{d}r$ at the origin). In this case

$$\text{ROI} = \frac{\mathrm{e}^r(\mathrm{e}^r - 1)}{\mathrm{e}^r - 1} - 1 = \mathrm{e}^r - 1, \tag{2.25}$$

which for positive values of r always exceeds r. This result is due to the fact that the time value of money is not taken into account in calculating ROI. It is evident that the case ROI $< r$ cannot occur in the case $n = 1$.

Within the range $0 < r < r_0$, ROI is less than r (i.e. $g(r) < 0$). The interval $(0, r_0)$ increases with n until it achieves its maximum at the value $n = 4$ (where $100r_0$ equals 49.9%), and decreases after this point. At the limit, as n approaches infinity, the interval $(0, r_0)$ shrinks into a single point (the origin). With larger values of r than r_0, ROI is larger than r.

Thus, the difference $g(r)$ is negative within the range $0 < r < r_0$, and its largest absolute value is reached at the point $r = r_m$. The dependence of this point on n is similar to the case of r_0 analysed above. First, r_m increases with n, decreases from the value $n = 4$ onwards, and approaches its limit, 0.

Table 1 also presents the absolute values of the difference g_m which correspond to the listed values of r_m. This difference follows the same behavioural pattern as the values of r_0 and r_m. The largest difference is attained at the value $n = 5$.

2.3. ROI as a function of IRR, the case of rising prices

2.3.1. Analytical discussion

The expressions for ROI that were derived in the previous section for different combinations of internal rates of return and rates of inflation are summarized below:

$$\text{ROI} = \begin{cases} \dfrac{1}{n}\left[\dfrac{R^n(S^n - 1)(R - S)}{(S - 1)(R^n - S^n)} - 1\right], & \text{when } r \neq s \neq 0, \\ \dfrac{1}{n}\left[\dfrac{R(R^n - 1)}{n(R - 1)} - 1\right], & \text{when } r = s \neq 0, \\ \dfrac{1}{n}\left[\dfrac{nR^n(R - 1)}{R^n - 1} - 1\right] & \text{when } r \neq s = 0, \\ 0, & \text{when } r = 0. \end{cases} \tag{2.26}$$

Inflation is taken into account by assuming that the cash inflows from the investment grow nominally at the rate of inflation, s. The real annual cash inflow stays constant.

Let us now analyse the difference function $g(r) = \text{ROI}(r) - r$ in the case where the rate of inflation is positive. The functions has two parameters, n and s. The aim of this exercise is to demonstrate that the behaviour of the $g(r)$-function does not qualitatively differ from the case of stable prices. If r, the internal rate of return, equals zero, the expression for the difference may be written as

$$g(0) = G(1) = \frac{1}{n}\left[\frac{1^n(S^n - 1)(1 - S)}{(S - 1)(1 - S^n)} - 1\right] - \ln 1 = 0, \tag{2.27}$$

while the other extreme value of $g(r)$ is obtained if r is allowed to approach infinity:

$$\lim_{r\to\infty} g(r) = \lim_{R\to\infty} G(R) = \lim_{R\to\infty}\left\{\frac{(S^n - 1)(R - S)}{n(S - 1)\left(1 - \frac{S^n}{R^n}\right)} - \frac{1}{n} - \ln R\right\} = \infty. \tag{2.28}$$

The general behaviour of $g(r)$ may be analysed using differentiation in the same manner as in the case of stable prices. The critical internal rate of return, r_0, is now, however, not only a function of the service life (n) but also a function of the rate of inflation (s). The general form of the derivative $\mathrm{d}G/\mathrm{d}R$ is

$$\begin{aligned}\frac{\mathrm{d}G}{\mathrm{d}R} &= \frac{\mathrm{d}}{\mathrm{d}R}\left\{\frac{(S^n - 1)}{n(S - 1)}\,\frac{R^{n+1} - SR^n}{R^n - S^n} - \frac{1}{n} - \ln R\right\}\\ &= \frac{(S^n - 1)\left[R^{2n+1} - (n+1)S^nR^{n+1} + nS^{n+1}R^n\right] - n(S - 1)(R^n - S^n)^2}{nR(S - 1)(R^n - S^n)^2}.\end{aligned} \tag{2.29}$$

In the origin of the $(r, g(r))$ space it may be expressed as

$$\begin{aligned}\left[\frac{\mathrm{d}g}{\mathrm{d}r}\right]_{r=0} &= \left[\frac{\mathrm{d}G}{\mathrm{d}R}\right]_{R=1}\\ &= \frac{(S^n - 1)\left[1 - (n+1)S^n + nS^{n+1}\right] - n(S - 1)(S^n - 1)^2}{n(S - 1)(S^n - 1)^2}\\ &= \frac{-S^n + nS - n + 1}{n(S - 1)(S^n - 1)}.\end{aligned} \tag{2.30}$$

The case $n = 1$ is, once again, exceptional. The expression for the derivative $\mathrm{d}g/\mathrm{d}r$ is then

$$\left[\frac{\mathrm{d}g}{\mathrm{d}r}\right]_{r=0} = \frac{-S + S - 1 + 1}{(S - 1)(S - 1)} = 0 \tag{2.31}$$

and the function itself becomes

$$G(R) = \frac{R(S - 1)(R - S)}{(S - 1)(R - S)} - 1 - \ln R = R - 1 - \ln R, \tag{2.32}$$

or, in terms of r

$$g(r) = \mathrm{e}^r - 1 - r. \tag{2.33}$$

This is always positive as long as $r > 0$.

In general (i.e. if $n \geqslant 2$), the following holds:

$$\begin{aligned}\left[\frac{\mathrm{d}g}{\mathrm{d}r}\right]_{r=0} &= \frac{-S^n + nS - n + 1}{n(S - 1)(S^n - 1)}\\ &= -\frac{(S - 1)^2\left[S^{n-2} + 2S^{n-3} + 3S^{n-4} + \cdots + (n-3)S^2 + (n-2)S + (n-1)\right]}{n(S - 1)(S^n - 1)}\end{aligned}$$

$$= -\frac{(S-1)^2 \sum_{t=0}^{n-2} (t+1) S^{n-2+t}}{n(S-1)^2 (S^{n-1} + S^{n-2} + \cdots + S + 1)}$$

$$= -\frac{\sum_{t=0}^{n-2} (t+1) S^{n-2-t}}{n \sum_{t=0}^{n-1} S^t} < 0. \tag{2.34}$$

The function $g(r)$ behaves thus in a similar manner as in the situation where prices are stable. It begins in the origin, is first negative and decreasing, becomes increasing at a point $r = r_m$, and positive when r exceeds r_0, growing finally towards infinity.

The critical internal rate of return, r_0, may be determined from

$$g(r) = \frac{e^{rn}(e^{sn} - 1)(e^r - e^s)}{n(e^s - 1)(e^{rn} - e^{sn})} - \frac{1}{n} - r = 0. \tag{2.35}$$

The solution value is a function both of the service life, n, and of the rate of inflation, s, and must be located numerically.

The internal rate of return, r_m, which corresponds to the largest negative value of $g(r)$, g_m, can be solved for from

$$\left[\frac{dg}{dr}\right]_{r=r_m} = \left[\frac{dG}{dR}\right]_{R=R_m} = 0, \tag{2.36}$$

or, alternatively, from

$$(S^n - 1)\left[R^{2n+1} - (n+1)S^n R^{n+1} + nS^{n+1}R^n\right] - n(S-1)(R^n - S^n)^2 = 0, \tag{2.37}$$

which, once again, must be solved numerically. For example, if the rate of inflation is 9.53% ($S = 1.10$), the values of r_m and r_0 for an investment with a length of life of ten years are 0.105 (10.5%) and 0.205 (20.5%), respectively.

2.3.2. Numerical discussion

The relationships between ROI and IRR were also examined with numerical analysis. ROI and the difference function $g(r)$ were calculated using different combinations of n and s, and letting the internal rate of return vary within the range 0–100%. The ranges of s and n were, respectively, 0–50% and 5–40 years.

The results confirmed the typical behaviour of $g(r)$ as derived earlier analytically, (i.e. $g(0) = 0$, $g(r) < 0$ as $0 < r < r_0$, $g(r_0) = 0$, $g(r) > 0$ as $r > r_0$). The effects of varying the rate of inflation are demonstrated in Fig. 2. ROI increases with the rate of inflation, which makes the function $g(r)$ move upwards. It is worth noting that, in the case of geometrically increasing net cash flows, Merrett and Sykes come to a conclusion which can be interpreted similarly. Their conclusion is, however, based on only one fixed numerical example (Merrett and Sykes (1960, p. 105)). As a consequence, the points r_0 and r_m move towards the origin and the absolute value of g_m is reduced. For any given value of the internal rate of return, the difference between ROI and the nominal internal rate of return grows with the rate of inflation.

The impact of inflation on $g(r)$ increases as the economic life of the investment becomes longer. This is caused by the absence of discounting in the ROI method. For example, if the investment under examination has an expected economic life of 40 years, inflation increases the value of ROI to such an extent that its value as a profitability indicator becomes highly questionable. If the internal rate of return in this case equals 20% in nominal terms and the rate of inflation is also 20%, the value of ROI would yield a profitability measure of 1025%. In order to provide a reliable approximation for the internal rate of return when both the length of life and the rate of inflation are expected to be large in value, ROI should be calculated only when the internal rate of return is expected to be low. In addition to the length of the service life and the rate of inflation, the difference between ROI and the nominal internal rate of return is crucially dependent on the magnitude of the latter. For sufficiently low values of the internal rate of return, however, the difference is negative (i.e. ROI is less than the internal rate of return).

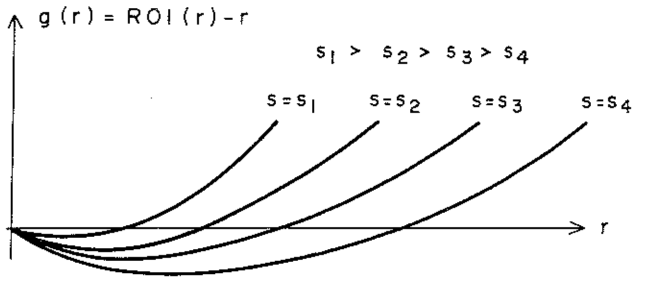

Fig. 2. The effect of inflation on the difference function $g(r)$.

2.3.3. Defining ROI on the basis of average invested capital

The previous discussions defined ROI using the initial capital expenditure as the denominator of the relevant ratio. This section analyses the relationships between ROI and IRR in the case where ROI is defined in relation to the average capital invested in the asset. This may be expressed as $[(n+1)/2n]\mathrm{NPV}(P)$, and ROI is defined by

$$\mathrm{ROI}=\frac{\bar{P}-\bar{D}}{\frac{n+1}{2n}\mathrm{NPV}(P)}=\frac{2}{n+1}\,\frac{\mathrm{SUM}(P)-\mathrm{NPV}(P)}{\mathrm{NPV}(P)}$$
$$=\frac{2}{n+1}\left[\frac{\mathrm{SUM}(P)}{\mathrm{NPV}(P)}-1\right]. \tag{2.38}$$

By substituting into (2.38) the expressions for SUM(P) and NPV(P), ROI can be expressed as (cf. expression (2.26))

$$\mathrm{ROI}=\begin{cases}\dfrac{2}{n+1}\left[\dfrac{R^n(S^n-1)(R-S)}{(S-1)(R^n-S^n)}-1\right], & \text{when } r\neq s>0,\\[2ex] \dfrac{2}{n+1}\left[\dfrac{R(R^n-1)}{n(R-1)}-1\right], & \text{when } r=s>0,\\[2ex] \dfrac{2}{n+1}\left[\dfrac{nR^n(R-1)}{R^n-1}-1\right], & \text{when } r>s=0,\\[2ex] 0, & \text{when } r=0.\end{cases} \tag{2.39}$$

The difference between ROI and the nominal internal rate of return (in terms of interest factor R) $G(R)=\mathrm{ROI}(R)-\ln R$ is now given by

$$G(R)=\frac{2}{n+1}\left[\frac{R^n(S^n-1)(R-S)}{(S-1)(R^n-S^n)}-1\right]-\ln R, \tag{2.40}$$

and the derivative $\mathrm{d}G/\mathrm{d}R$ becomes

$$\frac{\mathrm{d}G}{\mathrm{d}R}=\frac{2(S^n-1)}{(n+1)(S-1)}\,\frac{R^{2n}-(n+1)S^nR^n+nS^{n+1}R^{n-1}}{(R^n-S^n)^2}-\frac{1}{R}$$

$$=\frac{2(S^n-1)\left[R^{2n+1}-(n+1)S^nR^{n+1}+nS^{n+1}R^n\right]-(n+1)(S-1)(R^n-S^n)^2}{(n+1)(S-1)(R^n-S^n)^2}. \qquad (2.41)$$

When the nominal internal rate of return equals zero ($R=1$), the derivative is

$$\left[\frac{\mathrm{d}G}{\mathrm{d}R}\right]_{R=1}=\frac{2(S^n-1)\left[1-(n+1)S^n+nS^{n+1}\right]-(n+1)(S-1)(S^n-1)^2}{(n+1)(S-1)(S^n-1)^2}$$

$$=\frac{(n-1)S^{n+1}-(n+1)S^n+(n+1)S-n+1}{(n+1)(S-1)(S^n-1)}$$

$$=\frac{(S-1)^2\{(n-1)S^{n-1}+(n-3)S^{n-2}+\cdots+[n-1-2(n-1)]\}}{(n+1)(S-1)^2(S^{n-1}+S^{n-2}+\cdots+S+1)}$$

$$=\frac{\sum_{t=0}^{n-1}(n-1-2t)S^{n-1-t}}{(n+1)\sum_{t=0}^{n-1}S^t}. \qquad (2.42)$$

The denominator of (2.42) is clearly positive. The numerator contains an event number of powers of S. If the length of life, n, is an even number ($n=2m$, say), there are $n=2m$ terms in the numerator. If n is odd ($n=2m+1$), the coefficient of the power term in the exact middle becomes zero, and the remaining terms consitute an even number, $n-1=2m$. Half of the terms (the first m terms) have positive coefficients, which form the series $n-1$, $n-3,\ldots,3$, 1 if n is even and the series $n-1$, $n-3,\ldots,4$, 2 if n is odd. The rest of the terms (m last terms) have coefficients which, being arranged in descending order with respect to the powers of S, are -1, $-3,\ldots,-(n-3)$, $-(n-1)$, if n is even, or -2, $-4,\ldots,-(n-3)$, $-(n-1)$, if n is odd.

If all the coefficients, whose absolute values are equal, are paired, it is evident that the positive coefficient is always associated with a higher power of S, i.e. a larger positive number. Thus, the numerator as a whole, and consequently the whole expression, must be positive.

The function $g(r)$ is in the present case positive (ROI $> r$) from the value $r=0$ onwards, which differs from the analysis presented in Section 2.3.1, where $g(r)$ was found to be initially decreasing. This positive difference increases with the internal rate of return and finally approaches infinity. Although this result is difficult to prove analytically, it can be demonstrated numerically. Also, it becomes intuitively apparent, when the case, where ROI is based on the initial capital expenditure, is compared with the present case.

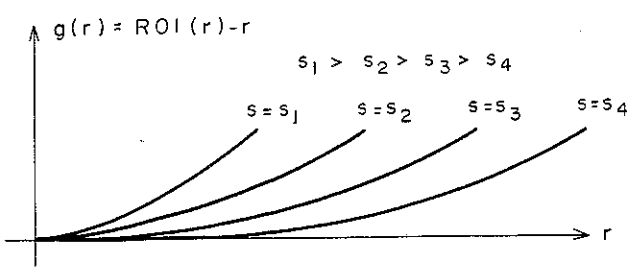

Fig. 3. The effect of inflation on the difference function $g(r)$.

Here, ROI is $2n/(n+1)$ times as large as the previous ROI at all levels of internal rate of return. This also implies that ROI overestimates IRR irrespective of the service life of the asset. This result is in accordance with that by Sarnat and Levy, the latter beeing derived, however, only in the case of stable prices (Sarnat and Levy (1969, p. 484)).

The difference $g(r)$ depends on the rate of inflation in a similar manner as has been discussed in Section 2.3.2, viz. increasing inflation moves the graph of $g(r)$ upwards. This is demonstrated in Fig. 3.

The graphs of $g(r)$ emanate from the origin and are monotonically increasing. The difference between ROI and the nominal internal rate of return is now always positive, and for a given nominal internal rate of return, the difference increases with the rate of inflation.

3. Summary and conclusions

This paper has analysed the relationships between ROI and IRR in the case, where the real cash inflows from the investment remain constant. If ROI is calculated using the initial capital expenditure as the denominator, the graph of the difference between ROI and the internal rate of return decreases from the origin as the internal rate of return becomes positive. Thus, the internal rate of return is higher than ROI for low values of the former. As the internal rate of return increases, the difference grows and reaches its maximum at a certain level (r_m) of the internal rate of return. After this, the graph of $g(r)$ is increasing, and intersects the r-axis (ROI $= r$, the internal rate of return) at a point r_0. The mathematical limit of $g(r)$ is in infinity.

Inflation reduces the distance between origin and the point r_0. The higher the rate of inflation, the faster ROI rises to the level of the internal rate of return. On the other hand, for a given rate of inflation (and $r > r_0$), the positive difference between ROI and r increases with the internal rate of return of the investment. The effect of inflation on increasing $g(r)$ is strengthened by lengthening the life of the investment in question. For example, if the internal rate of return in nominal terms is 20%, the expected length of life 40 years, and the rate of inflation 20%, the calculated value for ROI equals 1025%.

Even more generally, ROI overestimates the internal rate of return, if the expected service life of the project is long, to such an extent that it should not be used to indicate profitability.

The distorting effect of inflation on ROI is even more severe, if the profitability of the investment, as measured by its internal rate of return, is high.

It should be emphasized, however, that for sufficiently low values of r, ROI is lower than the internal rate of return thus underestimating the latter. For example, if the rate of inflation is 10%, and the expected length of life of the investment five years, ROI underestimated IRR up to the value of $r = 36.7\%$. If the length of life is increased to ten years, this underestimation occurs in the cases where the internal rate of return is 20% or less, and for an investment with an estimated service life of forty years, the relevant region ends with an internal rate of return of only 1.7%. Thus, ROI underestimates the internal rate of return of projects with a short expected length of life, and overestimates it for projects with longer expected service lives (see also Merrett and Sykes (1976, p. 210)).

If the average invested capital is used in the denominator of ROI instead of the initial capital expenditure, ROI is $2n/(n+1)$ times as large as in the previous case. The graph of the difference function ROI $- r$ still starts from the origin of the $(r, g(r))$ space, but never becomes negative. This means that ROI is always larger than the nominal internal rate of return, and this difference grows with the internal rate of return. ROI overestimates IRR irrespective of the length of the service life, and this overestimation increases with rising rates of inflation. Therefore, if ROI is based on the concept of average invested capital, its interpretation should be even more careful than in the previous case, especially if the projects under examination are very profitable and/or have long expected service lives.

These conclusions are valid even under conditions of stable prices, if the cash inflow from the investment grows at an annual rate of s (both in real and nominal terms). This results from the pre-tax approach adopted in this paper.

Appendix. List of symbols

$\bar{a}_{n\rceil r}$	present value factor for discrete payments where n is the number of periods and r the rate of discount,
D_t	depreciation in year t, $t = 1, 2, \ldots, n$,
$\bar{D}$	annual straight line depreciation,
$g(r)$	difference between ROI and the internal rate of return as a function of the latter, r,
g_m	largest negative value of $g(r)$ in absolute terms,
$G(R)$	difference between ROI and the internal rate of return as a function of R, the interest factor,
n	service life of the investment,
$\text{NPV}(P_t)$	present value of the cash inflow in year t in nominal terms; $t = 1, 2, \ldots, n$,
$\text{NPV}(P)$	present value of the sum of annual nominal cash inflows (years $1, 2, \ldots, n$),
P_0	real annual cash inflow from the investment (in the money of year 0),
P_t	nominal cash inflow in year t, $t = 1, 2, \ldots, n$,
$\bar{P}$	average annual nominal cash inflow,
r	internal rate of return on investment (continuous),
r_0	critical internal rate of return (equalises r and ROI),
r_m	internal rate of return which corresponds to the value of $g(r) = g_m$,
R	interest factor,
ROI	return on investment,
s	rate of inflation (continuous),
S	inflation factor,
$\text{SUM}(P)$	sum of the annual nominal cash inflows (years $1 - n$),
t	as a subindex: ordinal number of year, in e.g. discounting: number of years.

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