Optimal maintenance policy and planned sale date for a machine subject to deterioration and random failure

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Received September 1980
Revised February 1981

The decision problem concerning the optimization of the maintenance policy and the selection of the sale date for a machine subject to deterioration and random failure is considered from a control-theoretic viewpoint. The originally stochastic optimal control problem is converted to a deterministic optimal control problem with the coefficients of the state and control variables modified in the performance index. The maximum principle is applied to derive the conditions for the optimal maintenance policy and for the optimal planned sale date. Economic interpretations of these conditions are presented in terms of marginal costs and revenues. An explicit solution is found analytically for the problem in the special case when the failure probability is independent of maintenance. The case of exponentially distributed life time for the machine is analyzed in full detail. Finally, the results are illustrated by an example.

1. Introduction

Simultaneous determination of an optimal maintenance policy and sale date of a machine has been the subject of considerable recent investigation, both because of the inherent practical importance of the question and because of the interesting mathematical problems posed. From the point of view of mathematics, these problems fall naturally within the framework of optimal control theory. This approach to simultaneous maintenance–sale date optimization was initiated by Näslund [4] and followed up by Thompson [6], Arora and Lele [2] and Kamien and Schwartz [3].

By its functional forms the model formulated in [4] is a very general one. However, using Pontryagin's maximum principle, Näslund for the first time gives the general guidelines for optimizing the maintenance policy and sale date of the machine simultaneously.

The main contribution in Thompson's model [6] lies in making the functional forms given by Näslund explicit, so that it is possible to obtain both qualitative (the bang-bang type maintenance policy) and quantitative (the time point maintenance is switched off, the sale date of the machine) solutions. The model suggested by Arora and Lele [2] is a minor modification of Thompson's model. The type of the solution remains bang-bang although the number of qualitatively different solutions increases. Scott and Jefferson [5] have recently proposed a slightly modified bilinear model which provides a far richer spectrum of optimal maintenance policies than the linear models in [6] and [2].

While Thompson, Arora and Lele, and Scott and Jefferson consider the gradual deterioration of the machine with time only, and Kamien and Schwartz [3] the failure part (they take the probability of machine failure as the state variable) only, Alam and Sarma [1] have tried to incorporate both features in a single model. They use the model of Arora and Lele and incorporate the machine failure probability into the performance index by taking the life time of the machine as a random variable. The most serious

The author wishes to thank the referees for their helpful comments and suggestions. The financial support from the Finnish Entrepreneurs' Foundation is gratefully acknowledged.

0377-2217/82/0000–0000/$02.75

unsatisfactory aspect in their model is the fact that the sale date of the machine is not as an object of optimization: the machine is kept as long as it is operable, even though its use would not be profitable any more.

In the present paper we provide our own model, which extends and unifies the earlier results for the problem and represents a more realistic situation in practice. In Section 2 we incorporate the concept of planned sale date and its optimization into the model and thus remove the possibility of an unprofitable use of the machine which is included in the model [1]. In Section 3 we derive the necessary conditions for the optimal maintenance policy and planned sale date. We also show that these conditions have interesting economic interpretations in terms of marginal costs and revenues. In Section 4 we derive a detailed analytic solution for the problem in the special case, when the life time of the machine is exponentially distributed and independent of the maintenance. In Section 5 we work out a numerical example to illustrate the results and to compare the different models.

2. Formulation of the model

Arora and Lele consider the following problem: find the optimal maintenance policy $u^*(t)$ and the optimal sale date T for a machine to maximize the present value $V(T)$ of the machine given by

$$V(T) = S(T)\,\mathrm{e}^{-rT} + \int_0^T [pS(t) - u(t)]\,\mathrm{e}^{-rt}\,\mathrm{d}t, \tag{1}$$

where the salvage value $S(t)$ satisfies the relationship

$$\frac{\mathrm{d}S(t)}{\mathrm{d}t} = -a(t) - bS(t) + f(t)u(t), \quad S(0) = S_0. \tag{2}$$

In (1) and (2) r is the rate of interest, p is the (constant) production rate, $a(t)$ is the obsolescence rate, b is the (constant) depreciation factor and $f(t)$ is the maintenance effectiveness function. The maintenance expenditure function $u(t)$ is the control variable satisfying for all t, $0 \leq t \leq T$, the requirement

$$0 \leq u(t) \leq U. \tag{3}$$

The functions $a(t)$, $f(t)$ and $u(t)$ are assumed to be piecewise continuous and, in addition, $f(t)$ non-increasing. Using an exponential transformation (see [1, Eqs. (5) and (6)]), (1) and (2) can be changed to

$$V(T) = S_1(T)\,\mathrm{e}^{-(r+b)T} + \int_0^T \left[pS_1(t)\,\mathrm{e}^{-(r+b)t} - u(t)\,\mathrm{e}^{-rt}\right]\mathrm{d}t \tag{4}$$

and

$$\frac{\mathrm{d}S_1(t)}{\mathrm{d}t} = -a_1(t) + f_1(t)u(t), \quad S_1(0) = S_0, \tag{5}$$

where $S_1(t) = S(t)\,\mathrm{e}^{bt}$, $a_1(t) = a(t)\,\mathrm{e}^{bt}$ and $f_1(t) = f(t)\,\mathrm{e}^{bt}$. After this transformation, the solution of the problem can be found by a direct application of the maximum principle (see [1, p. 173]).

The model of Arora and Lele, given by (3) to (5), is the starting point of our analysis. However, the model considers only the gradual deterioration of the machine, and the possibility of machine failure is not taken into account. In the model given by Alam and Sarma the machine failure probability is taken into account by considering the variable T in the performance index (4) as a random variable. The problem thereafter becomes to find the optimal maintenance policy $u^*(t)$ so as to satisfy the conditions (3) and (5) and to maximize the expected value of $V(T)$, the expectation being taken with respect to the random life time of the machine. The problem is thus converted into a deterministic problem and can be solved via the usual maximum principle (see [1, pp. 173–174]).

In the model of Alam and Sarma, however, only the maintenance policy is optimized, and the selling of an operable machine is not considered. This may now give rise to serious consequences: in lack of the option of selling a still operable, but almost worthless and highly unprofitable machine, the model may lead to an unprofitable use of the machine and thus to an improper optimum for the problem. Therefore, we generalize the model and make it more realistic by taking also the sale date of the machine (called now the planned sale date due to the possibility of machine failing before that time) as a tool of optimization.

The state equation considered is (5) again. Let T denote the planned sale date of the machine, i.e. T is the time at which the machine will be sold provided it is still working. Let τ denote the life time of the machine and let $p_\tau(t; u(s),\ 0 \leqslant s \leqslant t)$, $P_\tau(t; u(s),\ 0 \leqslant s \leqslant t)$ and $Q_\tau(t; u(s),\ 0 \leqslant s \leqslant t) = 1 - P_\tau(t; u(s),\ 0 \leqslant s \leqslant t)$ denote the density function, the cumulative distribution function and the complementary distribution (or reliability) function, respectively, of the random variable τ. Further, let $p_\tau(t; u)$, $P_\tau(t; u)$ and $Q_\tau(t; u)$ compactly represent the preceding quantities. If τ gets a value less than T, the machine is junked at the time of its failure. The machine's junk value $J(t)$ obviously satisfies, for all t,

$$0 \leqslant J(t) \leqslant S(t). \tag{6}$$

Kamien and Schwartz assume [3, p. 496] the junk value $J(t)$ independent of the machine's age (as a constant J satisfying (6)), whereas Alam and Sarma take $J(t)$ equal to the salvage value $S(t)$. In order to make the two models, the model of Alam and Sarma and our own model, comparable and to clearly point out the effect of the sale date optimization, we follow here Alam and Sarma. The present value of the machine at such a time t, when the machine is still operable, is (cf. (4)):

$$V(t) = S_1(t)\,\mathrm{e}^{-(r+b)t} + \int_0^t \left[pS_1(s)\,\mathrm{e}^{-(r+b)s} - u(s)\,\mathrm{e}^{-rs} \right] \mathrm{d}s. \tag{7}$$

Let $V_0(T)$ be the present value really obtained, when the planned sale date is T. Obviously we have

$$V_0(T) = \begin{cases} V(T), & \text{if } \tau \geqslant T, \\ V(\tau), & \text{if } \tau < T. \end{cases} \tag{8}$$

The performance index to be maximized, the expected present value of the machine, is now

$$\begin{aligned} \mathrm{E}\{V_0(T)\} &= \int_0^T V(t) p_\tau(t;u)\,\mathrm{d}t + \int_T^\infty V(T) p_\tau(t;u)\,\mathrm{d}t \\ &= \int_0^T V(t) p_\tau(t;u)\mathrm{d}t + Q_\tau(T;u) V(T). \end{aligned} \tag{9}$$

Substituting (7) into (9) we get first

$$\begin{aligned} \mathrm{E}\{V_0(T)\} &= \int_0^T p_\tau(t;u) S_1(t)\,\mathrm{e}^{-(r+b)t}\,\mathrm{d}t + Q_\tau(T;u) S_1(T)\,\mathrm{e}^{-(r+b)T} \\ &\quad + \int_0^T \left\{ p_\tau(t;u) \int_0^t \left[pS_1(s)\,\mathrm{e}^{-(r+b)s} - u(s)\,\mathrm{e}^{-rs} \right] \mathrm{d}s \right\} \mathrm{d}t \\ &\quad + Q_\tau(T;u) \int_0^T \left[pS_1(t)\,\mathrm{e}^{-(r+b)t} - u(t)\,\mathrm{e}^{-rt} \right] \mathrm{d}t, \end{aligned} \tag{10}$$

which after integration by parts in the second integral and after some simplifications becomes

$$\begin{aligned} \mathrm{E}\{V_0(T)\} &= \int_0^T \left\{ \left[pQ_\tau(t;u) + p_\tau(t;u) \right] S_1(t)\,\mathrm{e}^{-(r+b)t} - Q_\tau(t;u) u(t)\,\mathrm{e}^{-rt} \right\} \mathrm{d}t \\ &\quad + Q_\tau(T;u) S_1(T)\,\mathrm{e}^{-(r+b)T}. \end{aligned} \tag{11}$$

Above we have (notationally) assumed throughout the derivation that the distribution of the machine life is continuous, i.e. the density function $p_\tau(t; u)$ exists. However, writing in (9) to (11) $\mathrm{d}P_\tau(t; u)$ instead of $p_\tau(t; u)\,\mathrm{d}t$ the formulae hold for distribution functions with discontinuities too. This is, of course, also true for the rest of the analysis.

From (11) we can readily see that it is of the same form as (4) in the corresponding deterministic model only with the coefficients of $S_1(t)$ and $u(t)$ modified. We can also note that (11) gives the performance index of the model by Alam and Sarma [1, Eq. (9)], if we in (11) set $T = \infty$. Thus the model of Alam and Sarma is got as a special case of our model with the planned sale date fixed to infinity (whereas our aim is to optimize T, too).

3. Analysis of the model

In the following we assume that the failure probability is not a function of maintenance. In this special case we can obtain explicit solutions for the problem and perform a detailed analysis of the model. Therefore, let $p_\tau(t)$, $P_\tau(t)$ and $Q_\tau(t)$ denote the density function, the cumulative distribution function and the reliability function which don't depend on maintenance. Then (11) becomes

$$\mathrm{E}\{V_0(T)\} = \int_0^T [p_1(t)S_1(t) - Q_\tau(t)u(t)] \,\mathrm{e}^{-rt}\,\mathrm{d}t + Q_\tau(T)S_1(T)\,\mathrm{e}^{-(r+b)T}, \tag{12}$$

where we have denoted $p_1(t) = [pQ_\tau(t) + p_\tau(t)]\,\mathrm{e}^{-bt}$ (cf. the notational comment on the distribution functions above).

Application of maximum principle gives the following optimal maintenance policy u^*:

$$u^*(t) = \begin{cases} U, & \text{if } Q_\tau(t)\,\mathrm{e}^{-rt} + \lambda(t)f_1(t) < 0, \\ \text{arbitrary} \in [0,U], & \text{if } Q_\tau(t)\,\mathrm{e}^{-rt} + \lambda(t)f_1(t) = 0, \\ 0, & \text{if } Q_\tau(t)\,\mathrm{e}^{-rt} + \lambda(t)f_1(t) > 0, \end{cases} \tag{13}$$

where $\lambda(t)$ is the adjoint variable and is given by

$$\lambda(t) = -Q_\tau(T)\,\mathrm{e}^{-(r+b)T} - \int_t^T [pQ_\tau(t) + p_\tau(t)]\,\mathrm{e}^{-(r+b)t}\,\mathrm{d}t. \tag{14}$$

Eq. (13) shows that the optimal maintenance policy is still bang-bang with any finite number of switches from U to 0 and/or vice versa (as we will later in the case of exponentially distributed life time demonstrate).

Next we will consider the economic interpretation of the formula (13) (for similar economic evaluations of some replacement models see [7]). For notational simplicity, let us assume that there exists exactly one switching point T' $(0 < T' < T)$ at which the maintenance is changed from U to 0. At the moment T' we have, see (13),

$$Q_\tau(T')U\,\mathrm{e}^{-rT'} + \lambda(T')f_1(T')U = 0. \tag{15}$$

Substituting (14) into (15) and rearranging terms, (15) can be written in the form

$$\begin{aligned} Q_\tau(T')U\,\mathrm{e}^{-rT'} &= \int_{T'}^T p_\tau(t)Uf(T')\,\mathrm{e}^{-b(t-T')}\,\mathrm{e}^{-rt}\,\mathrm{d}t + Q_\tau(T)Uf(T')\,\mathrm{e}^{-b(T-T')}\,\mathrm{e}^{-rT} \\ &\quad + \int_{T'}^T Q_\tau(t)pUf(T')\,\mathrm{e}^{-b(t-T')}\,\mathrm{e}^{-rt}\,\mathrm{d}t. \end{aligned} \tag{16}$$

On the left-hand side of (16), U is the (last) maintenance expenditure spent at moment T', $U\,\mathrm{e}^{-rT'}$ is the present value of this expenditure, and $Q_\tau(T')$ is the probability that this expenditure will be spent (the machine is still working at T'). With probability $P_\tau(T') = 1 - Q_\tau(T')$ on the other hand, no maintenance is needed at T'. The left-hand side of (16) can thus be interpreted as the expectation of the present value of the maintenance expenditure spent at the planned end of the optimal maintenance period. On the right-hand side of (16), $Uf(T')$ is the extra impulse in the salvage value of the machine yielding by the use of the maintenance expenditure U at moment T'. Further, $Uf(T')\,\mathrm{e}^{-b(t-T')}$ is the value of this impulse still remaining at time $t > T'$ (the depreciation rate of the salvage value is b), and $Uf(T')\,\mathrm{e}^{-b(t-T')}\,\mathrm{e}^{-rt}$ is its present value. We easily deduce that the right-hand side of (16) can be interpreted as the expected value of the profit, which the use of the maintenance expenditure U at moment T' yields on the present salvage value (the first two terms) and on the present value of the future operating receipts (the third term). The conclusion reached is thus: the planned ending point of the maintenance period (i.e. the moment maintenance is stopped for a still operable machine) has to be chosen so that the expected present value of marginal maintenance outlay is equal to its effect on the expectation of the discounted salvage value plus its effect on the expected cumulative cash flow. Analogical interpretations can be found if there are another number or another type of switches in the optimal maintenance policy.

Eq. (13) together with (14) gives the optimal maintenance policy with the planned sale date T considered as fixed. We still have to choose T so as to maximize $\mathbf{E}\{V_0(T)\}$. To do this we differentiate $\mathbf{E}\{V_0(T)\}$ in (12) with respect to T and set it equal to zero. Using similar reasoning as Thompson [6, p. 546] we get first

$$\frac{\mathrm{d}\mathbf{E}\{V_0(T)\}}{\mathrm{d}T}=\left[p_1(T)S_1(T)-Q_\tau(T)u^*(T)\right]\mathrm{e}^{-rT}+\left[\frac{\mathrm{d}Q_\tau(T)}{\mathrm{d}T}S_1(T)+Q_\tau(T)\frac{\mathrm{d}S_1(T)}{\mathrm{d}T}-(r+b)Q_\tau(T)S_1(T)\right]\mathrm{e}^{-(r+b)T}, \tag{17}$$

which after some labour becomes

$$\frac{\mathrm{d}\mathbf{E}\{V_0(T)\}}{\mathrm{d}T}=Q_\tau(T)\{\left[pS(T)-u^*(T)\right]+\left[f(T)u^*(T)-a(T)-bS(T)\right]-rS(T)\}\,\mathrm{e}^{-rT}. \tag{18}$$

At the optimum we thus have

$$\{pS(T)-u^*(T)\}+\{f(T)u^*(T)-\left[a(T)+bS(T)\right]\}=rS(T). \tag{19}$$

It is worth to note that the optimal planned sale date does not explicitly depend on the distribution of the life time. Its dependence on the life time distribution reveals via $u^*(T)$ which, of course, depends on $p_\tau(t)$. We can also obtain the following economic interpretation for this condition of optimality: the optimal planned sale date is reached at the moment when the net value of marginal operating receipts ($pS(T)-u^*(T)$) plus the net value of increase in the salvage value ($f(T)u^*(T)-[a(T)+bS(T)]$) are equal to the opportunity cost of the capital still invested at that moment ($rS(T)$). Further, it is easy to show that the condition (19) for the optimal planned sale date also gives the condition for the optimal sale date in the corresponding deterministic model by Arora and Lele (the sale date being there, however, actual).

4. A particular case: exponentially distributed life time

Eqs. (13) with (14) and (19) give us the general procedure to find out the optimal solution for the problem given in Section 2. We shall now demonstrate the explicit calculation of the optimal maintenance policy and planned sale date for a particular probability density function, viz. for the exponential case.

Let the constant failure rate of the machine be σ, whereafter (for $t \geqslant 0$)

$$p_\tau(t)=\sigma\,\mathrm{e}^{-\sigma t} \tag{20}$$

and

$$Q_\tau(t)=\mathrm{e}^{-\sigma t}. \tag{21}$$

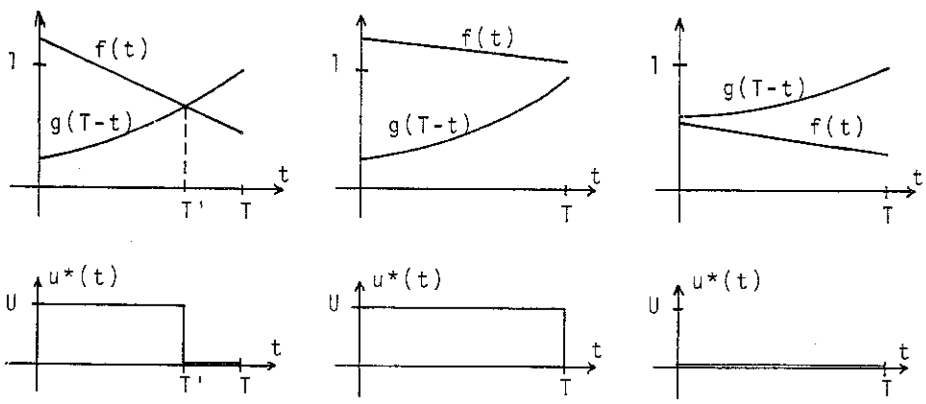

Fig. 1. Optimal maintenance policies for $p>r+b$.

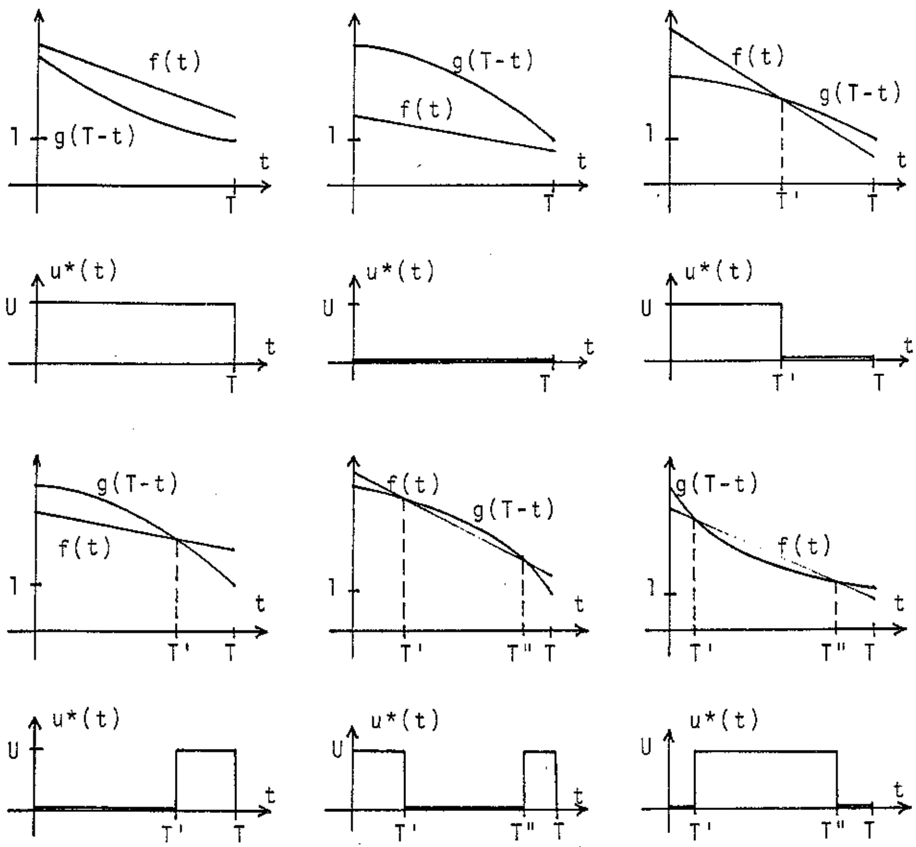

Fig. 2. Some optimal maintenance policies for $p<r+b$.

By substituting (20) and (21) into (14) and integrating we get for the adjoint variable $\lambda(t)$ the expression

$$\lambda(t) = -\frac{e^{-(r+b+\sigma)t}}{r+b+\sigma}\left\{p+\sigma-(p-r-b)\,e^{-(r+b+\sigma)(T-t)}\right\}. \tag{22}$$

Using (22) in (13), the following optimal maintenance policy $u^*(t)$ is obtained

$$u^*(t) = \begin{cases} U, & \text{if } f(t) > g(T-t), \\ \text{arbitrary} \in [0,U], & \text{if } f(t) = g(T-t), \\ 0, & \text{if } f(t) < g(T-t), \end{cases} \tag{23}$$

where

$$g(t) = \frac{r+b+\sigma}{p+\sigma-(p-r-b)\,e^{-(r+b+\sigma)t}}. \tag{24}$$

It is worth to note that the optimal maintenance policy (23) is of the same form as that of the corresponding deterministic model presented by Arora and Lele (cf. [1, Eq. (7)]). The only difference between these two optimal policies is in the expression of the function $g(t)$: in our stochastic model we must instead of the effective discount rate $r+b$ use the 'risk-adjusted' discount rate $r+b+\sigma$ and instead of the real production rate p use the 'risk-adjusted' production rate $p+\sigma$.

Next we shall consider the various forms which the optimal policy $u^*(t)$ may assume. We suppose, as before, that the maintenance effectiveness function $f(t)$ is a nonincreasing function of time. The nature of the function $g(t)$ depends on the values of the parameters p, r and b. We get three different cases depending on whether the production rate r is greater than or equal to or less than the effective discount rate $r+b$.

Case 1: $p > r + b$. Now $g(t)$ is a monotonically decreasing function of time. Fig. 1 shows the three qualitatively different possibilities for the optimal policy $u^*(t)$.

Case 2: $p = r + b$. Now we have $g(T-t) \equiv 1$, $0 \leqslant t \leqslant T$. The optimal policy $u^*(t)$ has one of the three forms given in Fig. 1.

Case 3: $p < r + b$. The function $g(t)$ is now monotonically increasing ($g(T-t)$ monotonically decreasing), otherwise it can be convex, concave or even *s*-shaped. This leads to various types of optimal maintenance policies. Some simple ones are presented in Fig. 2. One may also obtain more than two switching points where the level of maintenance changes.

5. An example

We shall now work out an example that illustrates both the similarities and differences between the models [2], [1] and our model. Suppose $p = 0.10$, $r = 0.05$, $a(t) \equiv 2$ (constant), $b = 0.03$, $\sigma = 0.04$, $S_0 = 100$, $U = 1$ and $f(t) = 1.5\ \mathrm{e}^{-0.02}$.

The model of Arora and Lele. As a basic reference we take the deterministic model of Arora and Lele, where the possibility of machine failure is not taken into account. Applying the criteria of the present model we obtain that the optimal sale date is $T = 5.3$ and the optimal maintenance policy is $u^*(t) = U = 1$ throughout the period of operation. The salvage value of the machine is

$$S(t) = 16.7\ \mathrm{e}^{-0.03t} + 150\ \mathrm{e}^{-0.02t} - 66.7, \quad 0 \leqslant t \leqslant 5.3, \tag{25}$$

and especially $S(5.3) = 82.5$. The maximal present value of the machine is $V(5.3) = 110.5$. The discounted profit from the use of the machine is $V(5.3) - S_0 = 10.5$.

The model of Alam and Sarma. The life time of the machine is now an exponentially distributed ($\sigma = 0.04$) random variable. The machine is kept as long as it is operable, the selling of an still operable machine is not considered. We get (see Eq. (13) in [1]) the following optimal maintenance policy

$$u^*(t) = \begin{cases} 1 & \text{for } 0 \leqslant t \leqslant 28.0, \\ 0 & \text{for } t > 28.0. \end{cases} \tag{26}$$

The salvage value (of an operable machine) proceeds as

$$S(t) = \begin{cases} 16.7\ \mathrm{e}^{-0.03t} + 150\ \mathrm{e}^{-0.02t} - 66.7, & 0 \leqslant t \leqslant 28.0, \\ 215\ \mathrm{e}^{-0.03t} - 66.7, & t > 28.0. \end{cases} \tag{27}$$

The expected present value of the machine, when the optimal maintenance policy (26) is followed, is (see [1, Eq. (9)]) $\mathbf{E}\{V_1(T)\} = 96.5$. We see that the expected profit $\mathbf{E}\{V_1(T)\} - S_0 = -3.5$ becomes negative, the use of the machine is unprofitable. In the following we shall show that by taking also the (planned) sale date as an object of optimization (the model of Section 2) the use of the machine can be made profitable.

The model with random failure and simultaneous optimization of maintenance and planned sale date. Now we consider our own model presented in Sections 2 to 4. Using the criteria derived for the case of an exponential life time we obtain the optimal planned sale date $T = 5.3$ and the optimal maintenance policy $u^*(t) = 1$, $0 \leqslant t \leqslant T = 5.3$. The salvage value of the machine proceeds, provided the machine has not failed, according to (25). The expected present value of the machine is $\mathbf{E}\{V_0(T)\} = 101.1$, thus the expected profit becomes $\mathbf{E}\{V_0(T)\} - S_0 = 1.1$. We see that optimization of the sale date also in the case of a stochastically failing machine is necessary: instead of the negative expected profit -3.5 (when the machine is kept as long as it is operable; the model of Alam and Sarma) we get a positive result 1.1 (when the machine is kept only until $T = 5.3$ or until it fails, whichever comes first). However, the expected profit ($= 1.1$) is less than the profit in the corresponding deterministic case ($= 10.5$) due to the possibility of a failure before the optimal sale date. As a conclusion, the example clearly points out the importance of the sale date optimization also in the case of random life time of the machine.

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