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Non-linear Advertising Capital Model with Time Delayed Feedback Between Advertising and Stock of Goodwill

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Abstract—According to the classical Nerlove–Arrow model, advertising expenditure can be considered as a capital investment to create present and future demand for the firm's products and, hence, to create present and future revenues for the firm. Advertising is assumed to influence via stock of goodwill which cumulatively counts for the effects of the firm's current and past advertising outlays. The paper presents a time delayed feedback model describing the relations between advertising and goodwill. Three different types of effects of advertising upon the dynamics of goodwill are modelled. The advertising policy of the management is incorporated into the model via a non-linear advertising function. The advertising function controls the advertising outlay e.g. by budget constraint and by the actual and target values of goodwill. The behavior of the model is analysed both analytically and numerically. Special attention is given for deriving the stability conditions for the limiting solution. The cases of repelling or chaotic limiting solutions are analysed by bifurcation and state space diagrams. Several numerical examples are given. Copyright © 1996 Elsevier Science Ltd.

Motto:

"Doing business without advertising is like winking at a girl in the dark. You know what you are doing, but nobody else does."

(Steuart Henderson Britt, New York Herald Tribune, 30.10.1956, according to Colin Dexter in 'The Way Through the Woods'.)

1. INTRODUCTION

This study is based on the classical Nerlove–Arrow model; its purpose is to study the behavior of goodwill defined in accordance with the classical model [1]. The classical model and all its optimal control theory extensions are optimization models. In the state space approach of the control theory the purpose is to find the optimal control (see e.g. Ref. [2]). Sethi [3] made a comprehensive survey of the literature on dynamic optimal control models in advertising devoted to determining optimal advertising expenditures over time subject to some dynamics that define how advertising expenditures translate into sales and in turn, into profits for a firm or a group of firms under consideration. The survey by Sethi was organized in four model categories which are no longer sufficient. Therefore a new, broader classification was developed [4]. The Nerlove–Arrow model belongs in this classification to the category of capital stocks generated by advertising, price and quality.

The behavior of goodwill, without optimization, was studied by Ostrusska [5] in an interesting way. In that model, where goodwill was chosen as a state, the development of goodwill was studied by establishing a feedback system between advertising and goodwill and introducing the time lag effect of advertising on goodwill. Ostrusska's model was a nonlinear dynamical system exhibiting both periodic and chaotic behavior. Another way to study the nonlinear economic relations is based on the analysis of empirical time series (see e.g. Refs [6, 7]).

The aim of the present paper is to study the suitability of chaos theory for clarifying the behavior of advertising models used in marketing studies. Within this mathematically oriented framework the aim of this study is, on the one hand, to study the behavior of goodwill in relation to different advertising policies, and, on the other hand, to study the different effects of advertising on the dynamics of goodwill. The research plan is to study three different advertising effects which lead to nonlinear transient dynamics, and, on the other hand, can lead the system to an equilibrium. Starting from the Nerlove–Arrow model, a time delayed feedback system between advertising and goodwill is developed. It is assumed that the effects of advertising on the equilibrium goodwill can be linear, nonlinear or bounded.

An essential part in studying the development of goodwill is to determine the conditions for the stability of the limit point of the system. In the case of monopoly, it is important for a firm that the behavior of its goodwill has a fixed point attractor. If the behavior of goodwill is either periodic or chaotic, the firm has difficulties in estimating its goodwill as well as in planning its advertising strategy. The stability condition for a limit point can be determined analytically when the time delay (τ) is equal to zero or one. So analytical methods are used in the case of the immediate effect of advertising ($\tau = 0$) and in the case of a short delay effect of advertising ($\tau = 1$). In the case of longer delay effects of advertising ($\tau > 1$) we must be content with simulation.

As a result the ideas of linear, nonlinear and bounded effects of advertising lead to a very interesting and reasonable behavior of goodwill. The linear model is simple, but it doesn't take the increasing marginal cost of goodwill or a saturation level for equilibrium goodwill into consideration. These ideas can be taken into account in the nonlinear and in the bounded model. If the depreciation rate of goodwill is small and there is no external disturbance the behavior of the model tends to a stable state. If the depreciation rate of goodwill is too high or a firm is too elastic in decreasing its advertising after the target has been reached, periodic or chaotic behavior of goodwill may follow as well as if the time delay of advertising is too long.

The paper is organized as follows. Section 2 presents the Nerlove–Arrow model and the ideas of linear, nonlinear and bounded effects of advertising. The market behavior is derived using first the assumption of constant advertising. Then it is allowed that advertising may vary. The advertising policy of the management is incorporated into the model via a nonlinear advertising function, and the stability conditions of equilibrium are derived with the help of the goodwill elasticity of advertising at the target (i.e. at the equilibrium). In Section 3 the model is analyzed in the linear, in the nonlinear and in the bounded case. The paper ends with a summary in Section 4.

2. THE MODEL

2.1. *The Nerlove–Arrow model*

Advertising expenditure is in many ways similar to investments in durable plant and equipment. The latter affects the present and future net revenue of the investing firm. Advertising expenditures on their part affect the present and future demand for the product and, hence, the present and future net revenue of the firm advertising [1]. One way to present the temporal differences of the effects of advertising on the demand is to define a stock which is called goodwill and denoted by $g(t)$. Goodwill is supposed to summarize the effects of current and past advertising outlays on demand. If the price of a unit of goodwill is $1, then a dollar spent on advertising increases goodwill by an equal amount. On the other hand, a dollar spent some time ago should, according to the previous argument, contributes less. One possible way to present this lesser contribution is to say that goodwill depreciates. If it is further assumed that current advertising expenditure cannot be negative and that depreciation occurs at a constant proportional rate, r, we get the equation

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t} + rg(t) = a(t) \geqslant 0, \tag{1}$$

where $a(t)$ is current advertising outlay and $a(t)$ and $g(t)$ are continuous functions of time. Equation (1) states that the net investment in goodwill is the difference between the gross investment (current advertising outlay) and the depreciation of the stock of goodwill [1].

The Nerlove–Arrow Equation, eqn (1) assumes that there is no time lag between advertising expenditures and increases in the stock of goodwill. As the demand for the monopolist's product is a function of the stock of goodwill, that implies that the rate of sales at time t totally adjusts itself immediately to the rate of advertising prevailing at time t [8]. Introducing a time lag between the rate of advertising and its effect on the rate of sales leads to a control problem in which the equation of motion is given by an integro-differential equation. There are many generalized Nerlove–Arrow models dealing with distributed time lags in this way (e.g. [8, 9–11]).

If it is assumed that the change of goodwill at time t depends on the amount of advertising in an earlier period a certain time ago, a fixed time delay τ can be incorporated into eqn (1) [5]

$$\frac{\mathrm{d}g(t)}{\mathrm{d}t} = a(t - \tau) - rg(t). \tag{2}$$

In the previous models with time delays it had been assumed that advertising was independent of sales. When the advertising budget was, however, assumed to depend on sales, which, on the other hand, depended on the goodwill stock, the amount of advertising could be described as a function of goodwill [5].

By choosing different values for a in eqn (1) or (2), many different advertising policies can be incorporated into the model. Ostrusska's idea was to use an advertising function of the following type:

$$a(g) = bge^{-(g/m)^2}, \tag{3}$$

where b and m are parameters and $g = g(t)$ describes the development of goodwill as a function of time. Luhta [12] has presented several generalized versions for the advertising function, eqn (3). The different advertising policies are discussed in more detail in Section 2.3.

2.2. *Market behavior: the effects of advertising on the dynamics of goodwill*

In the classical Nerlove–Arrow model it is assumed that a dollar spent on advertising increases goodwill by an equal amount of money. It is assumed, in other words, that the marginal cost of goodwill is constant. This linear relation can be motivated, for example, via the model, equation (1), as follows.

Let us assume, for the moment, that advertising is held constant all the time. Take the discrete formulation of model (1):

$$g_{t+1} = g_t - rg_t + a \tag{4}$$

and consider the behavior of the system towards the equilibrium. It is easy to see that the equilibrium state is stable and the equilibrium level of goodwill is

$$g^* = \frac{a}{r}. \tag{5}$$

This means that the equilibrium goodwill g^* is linearly related to advertising as assumed in Nerlove–Arrow model. The marginal cost of goodwill is, therefore, constant in equilibrium. It can be further seen that the equilibrium goodwill is inversely proportional to the depreciation rate r of goodwill.

The linear relation does not seem very reasonable in practice and it is more reasonable to assume that the marginal cost of goodwill is increasing. This can be motivated, for example, with the forgetting effect of the system. The increasing marginal cost of goodwill can be incorporated into the model by adding an exponent $q > 1$ to the depreciation term

$$g_{t+1} = g_t - r g_t^q + a. \tag{6}$$

This means nonlinearity of the model at the same time. The equilibrium goodwill is now

$$g^* = \left(\frac{a}{r}\right)^{1/q}. \tag{7}$$

One natural approach is to assume that there is a saturation level for the equilibrium goodwill. This can be modelled by adding an upper bound G for the goodwill in model (1). The transient dynamics of the system is then

$$g_{t+1} = g_t - r g_t + a\left(1 - \frac{g_t}{G}\right). \tag{8}$$

The equilibrium goodwill from model (8) is

$$g^* = \frac{aG}{rG + a}. \tag{9}$$

Equation (9) also means increasing marginal cost for goodwill.

It has been assumed above that the advertising outlay is constant along time. The assumption was made only for finding a way to model the effect of markets on the development of goodwill. Three different effects were considered: the linear effect, eqns (4) and (5), the nonlinear effect, eqns (6) and (7), and the bounded case, eqns (8) and (9). Figure 1 depicts the three market effects. In the following, we give up the assumption of constant advertising and consider the system under non-constant advertising environment via different advertising policies.

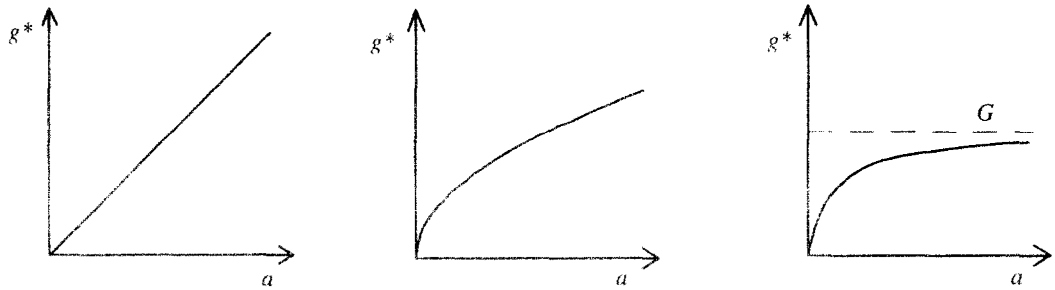


Fig. 1. (a) Linear effect; (b) nonlinear effect; (c) bounded effect.

2.3. *Advertising policies of the management*

As a starting point in modelling advertising policies we take model (3) by Ostrusska. Figure 2 presents a schematic description of the model. In the beginning, goodwill increases with the help of advertising and the management can increase advertising to the maximum possible amount of money ($= \bar{a}$) allowed by the chosen policy. The larger the value of the scale parameter b is the larger is the advertising budget and the value of $\bar{a}$. Furthermore, a target goodwill is assumed to exist and is denoted by g^*. The corresponding advertising expenditure is a^*. (The target goodwill is denoted by g^* because in the following the analysis will be concentrated on equilibrium goodwills only.)

The maximum amount of money $\bar{a}$ to be spent on advertising is reached when the value of goodwill is estimated to be $\bar{g}$, i.e. before the target goodwill has been reached. Following model (3), the maximum amount of advertising expenditure $\bar{a} = bm2^{-1/2}\,e^{-1/2}$ is used, when the estimated goodwill is $\bar{g} = 2^{-1/2}m$. We see, therefore, that m is a location/shape parameter of the model.

It is natural to assume that in the beginning goodwill is smaller than the target goodwill, $g_0 < g^*$. So for a start, a large amount of money is spent on advertising and before the target value of goodwill will be reached, the highest amount of money allowed by the budget has to be spent on advertising. After the advertising outlay has reached its peak it starts to decrease. So far the advertising function of Ostrusska seems to function rather well because it contains the above requirements. A more easy-to-handle form of the advertising function (3) would, however, be

$$a(g) = bg\,e^{-(1/2)(g/m)^2}, \tag{10}$$

where the maximum advertising expenditure $\bar{a} = bm\,e^{-1/2}$ is used when goodwill is estimated to be m. A still easier form for interpreting the parameters is the model:

$$a(g) = bg\,e^{(1/2)(1-(g/m)^2)}, \tag{11}$$

where the maximum advertising expenditure $\bar{a} = bm$ is used when goodwill is m.

To develop the model further, it is natural to assume that investment in advertising will not be ceased abruptly but, on the contrary, the firm continues to advertise to get a better and better image. Then a general exponent s, instead of the constant 2, can be incorporated into function (11) to generate this property. In this case the advertising function becomes [12]:

$$a(g) = bg\,e^{(1/s)(1-(g/m)^s)}, \tag{12}$$

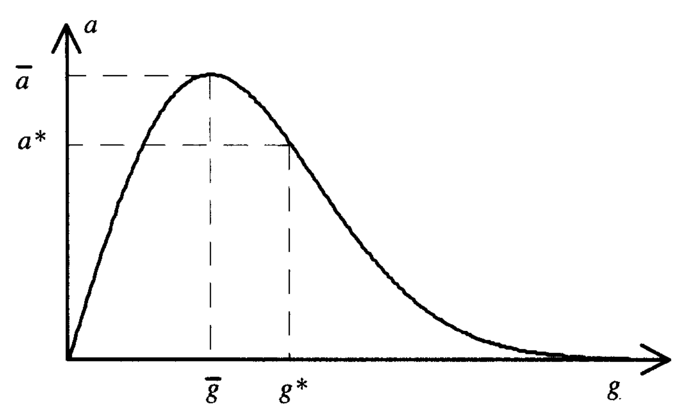


Fig. 2. Dependence of advertising budget on goodwill.

where the advertising expenditure $\bar{a} = bm$ when goodwill $\bar{g} = m$. It is interesting to note that the parameter s has no effect on the maximum point and the maximum value of the function (12).

Using formula (12), there will be possibilities for different advertising strategies, and the management can also look for an optimal advertising with it. In Fig. 3 a nonlinear market effect is assumed to exist (the curve $a^* = r(g^*)^q$). In the case of a stable equilibrium, the target goodwill g^* is maintained with the advertising outlay a^*. Advertising policies a_1–a_4 represent the family of different advertising policy to reach the target goodwill g^*. The policy to be used is chosen by fixing the values for the parameters b, m, and s. If there exists a fixed upper bound for the advertising budget (a_{max} in the figure), some of the policies may be infeasible (a_4 in the figure).

Function (12) can be regarded as a reaction function for the estimated goodwill. When the parameter s is large the advertising flow $a(g)$ decreases quickly after the value m and when the parameter s is small the advertising flow decreases slowly after the value m. Setting the value for the parameter s is a long-term decision in a firm, so it is important that the management defines the advertising policy of the firm in the right way when attempting to attain more permanent goodwill for the firm.

2.4. *Stability conditions*

If advertising is not constant, the stability of equilibrium depends on the local properties of the advertising function, which, on the other hand, is a function of goodwill. These local properties can be described in terms of the goodwill elasticity of advertising at the target

$$\varepsilon = \frac{g^*}{a^*}\frac{\mathrm{d}a}{\mathrm{d}g}\bigg|_{g=g^*} = \frac{g^* a'(g^*)}{a(g^*)}, \tag{13}$$

where g^* is equilibrium goodwill which is supposed to be equal to the target goodwill. Now the previously mentioned three effects of advertising act as bases for the three different approaches in an attempt to study the stability conditions of the limit point of the system.

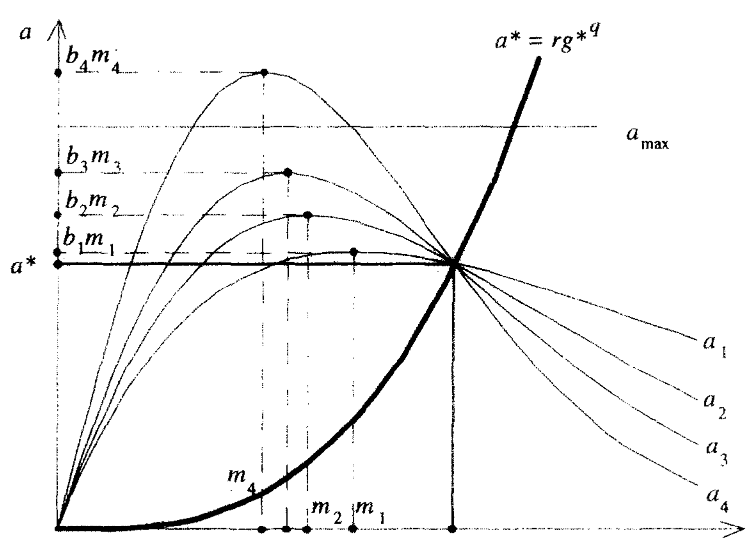


Fig. 3. The family of different advertising policies to reach the stable target value g^* for goodwill (nonlinear market effect).

In economics it is common to use elasticity when describing the properties of the neighbourhood of equilibrium. The same local properties, in the theory of dynamic systems, can be studied by means of the definition of the stability conditions of the equilibrium below [13]. This can be done by defining $g_{t+1} = F(g_t)$ and studying the properties of the function F. The limit point g^* of the function F is stable if

$$|F'(g^*)| < 1. \tag{14}$$

The stability condition, eqn (14), can be interpreted geometrically as follows. As an approximation for function $g_{t+1} = F(g_t)$ we first have (near the point g^*) $F(g_t) - g^* \approx F'(g^*)(g_t - g^*)$ or $g_{t+1} - g^* \approx F'(g^*)(g_t - g^*)$. For $F'(g^*)$ we thus have $F'(g^*) \approx (g_{t+1} - g^*)/(g_t - g^*)$. It is now intuitively clear that, for g^* being an equilibrium point, we must have $|g_{t+1} - g^*| < |g_t - g^*|$ or equivalently $|F'(g^*)| < 1$.

In the linear case the limit point of the function $F(g_t)$, i.e.

$$F(g_t) = g_t - rg_t + a(g_t), \tag{15}$$

is stable if

$$|F'(g^*)| = |1 - r + a'(g^*)| = |1 - r(1 - \varepsilon)| < 1. \tag{16}$$

The corresponding nonlinear effect of advertising leads to the stability condition

$$|F'(g^*)| = |1 - rq(g^*)^{q-1} + a'(g^*)| = |1 - r(g^*)^{q-1}(q - \varepsilon)| < 1. \tag{17}$$

In the bounded case the equation

$$\begin{aligned} |F'(g^*)| &= \left|1 - r + a'(g^*)\left(1 - \frac{g^*}{G}\right) - \frac{a(g^*)}{G}\right| \\ &= \left|1 - r(1 - \varepsilon) - \frac{rg^*}{G - g^*}\right| < 1 \end{aligned} \tag{18}$$

is obtained.

3. ANALYSIS OF THE MODEL

Next the behavior of goodwill is studied by means of three different versions of the model. The equations are based on three different market laws, i.e. on the linear, on the nonlinear and on the bounded effect of advertising. The reaction function for the estimated goodwill is eqn (12). The reaction of advertising is either immediate or time delayed. The analysis of the model concentrates especially on the examination of the conditions for the stability of the limit point of the system.

3.1. *The linear market effect*

The system with the linear market effect, eqn (4), and the advertising function, eqn (12), is

$$\begin{cases} g_{t+1} = g_t - rg_t + a(g_{t-\tau}) \\ a(g) = bg\,\mathrm{e}^{(1/s)(1-(g/m)^s)}. \end{cases} \tag{19}$$

When the time delay is zero, $\tau = 0$, the behavior of goodwill can be studied analytically by determining the stability conditions for a fixed point attractor.

3.1.1. Advertising with immediate effect. The equilibrium goodwill is

$$g^* = m\left(1 - s\ln\left(\frac{r}{b}\right)\right)^{1/s}. \tag{20}$$

From eqns (13) and (20), the elasticity becomes

$$\varepsilon = 1 - \left(\frac{g^*}{m}\right)^s = s\ln\left(\frac{r}{b}\right). \tag{21}$$

By joining eqns (16) and (21), the following stability condition for the limit point is obtained

$$\left|1 - r\left(\frac{g^*}{m}\right)^s\right| < 1. \tag{22}$$

The simplified form of this is

$$b < r\cdot e^{(2-r)/sr}. \tag{23}$$

This means that, when parameter r is small, there is no restrictions for parameter b. From the point of view of stability, problems may only appear if parameter r is too high (see Fig. 4). So, if the depreciation rate of goodwill is too high, a stable situation cannot be reached by increasing the volume of advertising to a great extent. For example, for parameter values $r = 0.5$ and $s = 1.5$ it must be $b < 0.5\,e^2 = 3.7$ in order to have a stable equilibrium point. A stable situation can be reached only when goodwill is decreasing slowly enough. So it looks as if it would be good for a firm to have loyal customers. One interesting point is to note that the stability condition, eqn (23), is independent of the location parameter m.

The stability condition for the limit point can also be presented in the form

$$r < \frac{2}{1 - \varepsilon}. \tag{24}$$

This means that when the depreciation rate of goodwill r is high, it is possible to reach the target g^*, i.e. a stable situation, by means of low absolute values of goodwill elasticity ε only (see Fig. 5). The elasticity must be very small in relation to the target in this situation. In the case of low depreciation rate, stability sets no requirements for the elasticity.

3.1.2. Advertising with lagged effect. When describing the long-term behavior of goodwill by means of time-lagged models, the systems become multidimensional. The only way

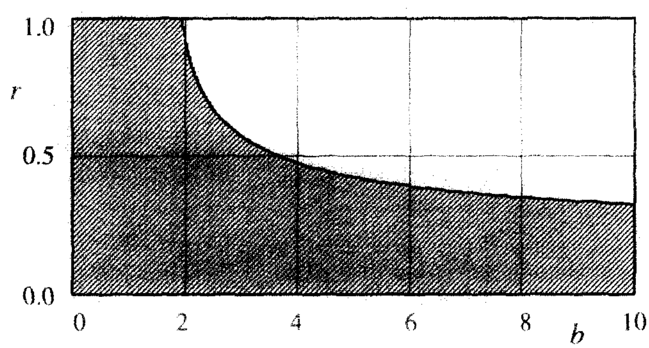


Fig. 4. Values for parameters r and b ($s = 1.5$) when the system has a fixed point attractor: the shaded area.

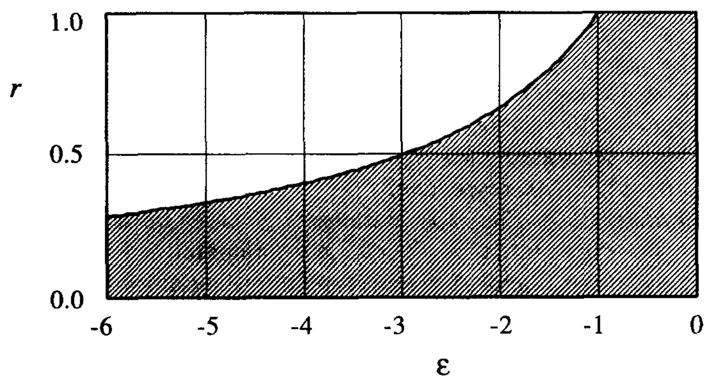


Fig. 5. Values for parameters r and ε when the system has a fixed point attractor.

now, when the time lag is greater than one, is to approach the behavior of goodwill using numerical methods. The development of the state of goodwill with different parameter values can be illustrated with the help of trajectories or with the help of attractors which have been developed by iteration [12]. A good description of the behavior of equilibrium goodwill can be achieved by using bifurcation diagrams with varying parameters. These are also produced by functional iteration.

When there is no lag in the dynamics, i.e. the parameter $\tau = 0$ in the model (19), the bifurcation diagram exhibits a period-doubling route to chaos. Figure 6(a) shows a highly simplified account of how the structure of the equilibrium behavior of the system (19) changes as the parameter r varies. This kind of behavior is typical to an one-dimensional model [13]. If $\tau > 0$, i.e. there is lag in the dynamics, the bifurcation diagram (see Fig. 6(b)) differs from the previous one because the model has now a higher dimension and the bifurcations are Hopf bifurcations [13, 15].

The next two figures, Fig. 7(a) and (b), illustrate the development of the equilibrium goodwill when a simultaneous variation of two different parameters (b and r) is considered. The figures have been produced using the period testing algorithm developed

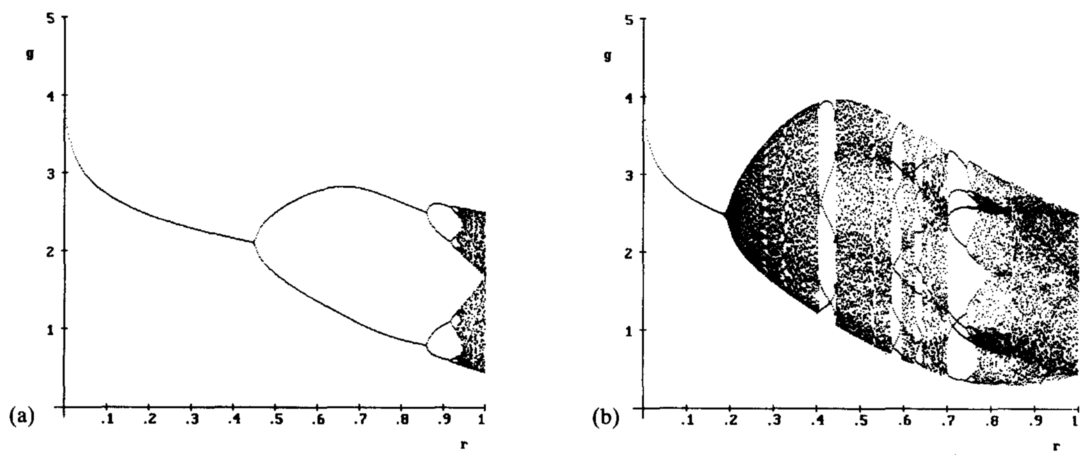


Fig. 6. The values of goodwill with different values of parameter r, when $b = 2.5$ is fixed, $s = 2$ and (a) $\tau = 0$; (b) $\tau = 1$.

by Laaksonen [14]. The algorithm colours the stable, periodic and chaotic regions with different colours in the phase space of the two variables. The green regions show the fixed point attractors, the white regions the chaotic attractors, and the other colours the periodic attactors up to seventy different periods. In Figure A1 of the Appendix the periods of Figure 7(b) have been separated into five subpictures which describe the periods 1–14, 15–28, 29–42, 43–56 and 57–70, respectively.

In the bifurcation diagram of Fig. 6(a), parameter r varies between 0 and 1 whereas the amount of advertising expenditure is fixed, i.e. parameter $b = 2.5$. In Fig. 7(a), also parameter b varies (between 0 and 5). The bifurcation diagram in Fig. 6(b) describes thus a special case of the family of systems presented in Fig. 7(b): parameter b is fixed to be equal to 2.5. Because the time lag is now zero, the period-doubling route to chaos can easily be seen. A similar figure has also been drawn in relation to the case when the time lag is one (see Fig. 7(b)). Comparing Fig. 6(a) and (b) on the one hand and Fig. 7(a) and (b) on the other hand, one easily observes that time lag is closely related to the dynamics of goodwill.

In the bifurcation point (at $r = 0.20$ in Fig. 6(b)), an invariant closed curve is born as the attracting fixed point becomes repelling (see Fig. 8). The vertical cut of the bifurcation diagram in Fig. 6(b) is a projection of the attracting set in Fig. 8. The visual inspection of the attracting set is referred to a return map [7]. When the value of goodwill at t ($= g_t$) is plotted against that at $t - 1$ ($= g_{t-1}$) a clear difference between the chaotic and the random behavior is seen. For the random time series no pattern emerges. The order of the attractor is not easily seen in the bifurcation diagram.

In the bifurcation diagrams, where a fixed advertising budget (parameter b) is assumed, it can be seen how the goodwill behavior varies with different values of the depreciation rate r. In the case of immediate effect of advertising ($\tau = 0$) the periodic behavior starts when $r \approx 0.45$. When $r < 0.45$ the system has a stable equilibrium goodwill g^* which is smaller the bigger r is. When $r > 0.45$ the equilibrium point jumps first between two different values, then between four different values of goodwill, and so on. In the case of short delay effect of advertising ($\tau = 1$) the cyclic behavior at the bifurcation point begins when $r \approx 0.20$. From the point of view of the assumption that there is a time lag effect of advertising the cyclic behavior is plausible. When the limit point of goodwill becomes unstable it is reasonable to assume that the behavior of goodwill becomes cyclic instead of going up and down between two values as in the case of immediate effect advertising.

The stability condition for the fixed point attractor can be determined analytically also in the case when the time lag $\tau = 1$ [13]. In this study it can be done with the help of the function

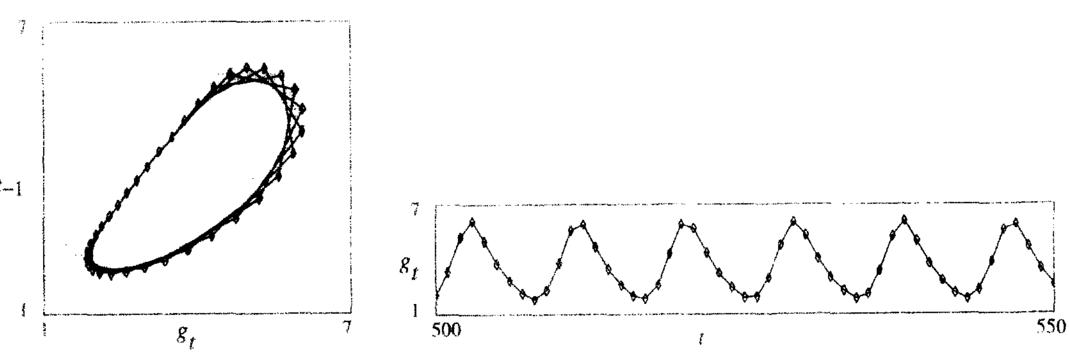


Fig. 8. The attracting set of the model and the time development of goodwill, when $b = 2.5$, $s = 2$, $r = 0.25$ and $\tau = 1$.

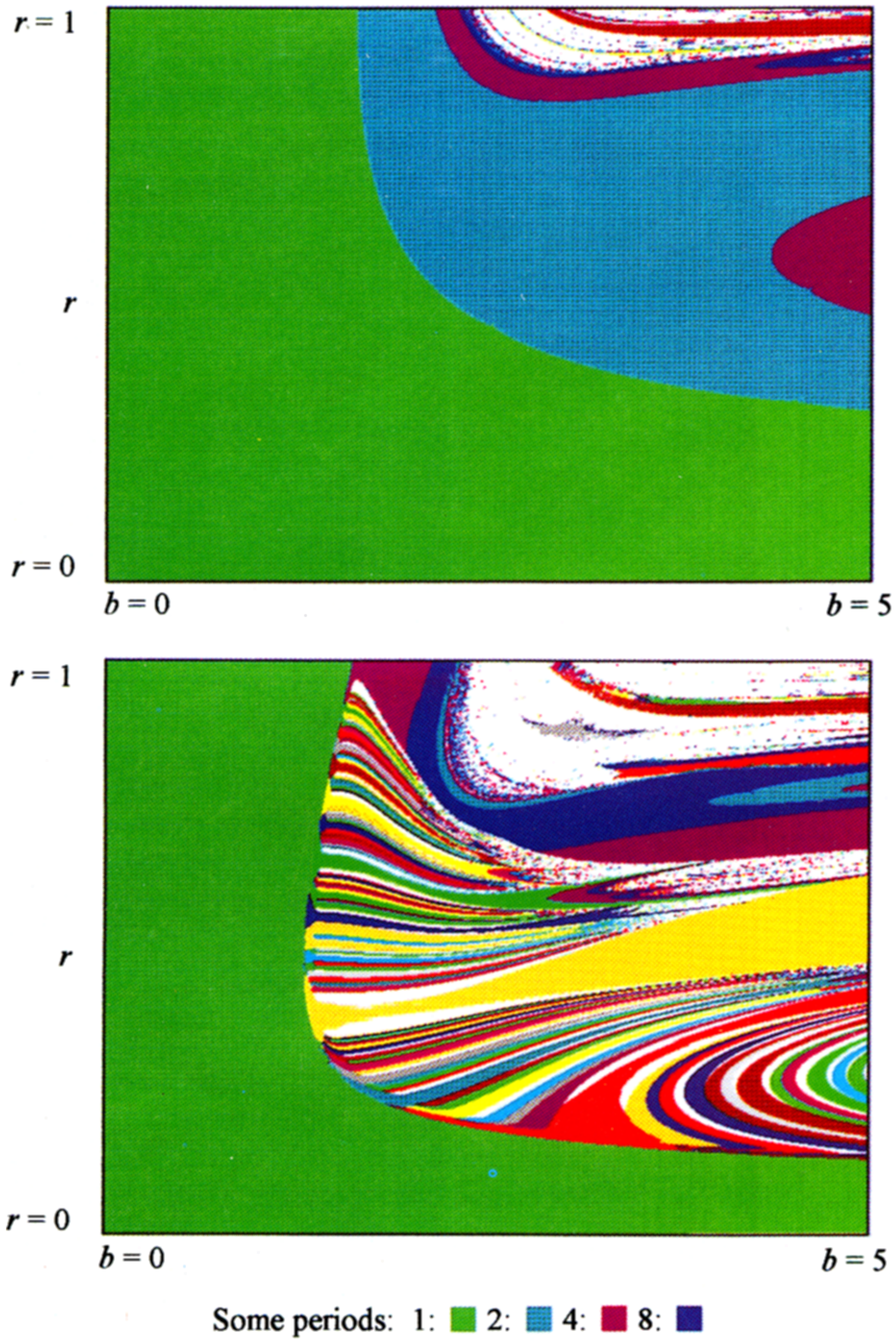


Fig. 7. (a) $\tau = 0$, $s = 2$, $0 \leqslant r \leqslant 1$, $0 \leqslant b \leqslant 5$; (b) $\tau = 1$, $s = 2$, $0 \leqslant r \leqslant 1$, $0 \leqslant b \leqslant 5$. For more detailed data for the periods, see the Appendix.

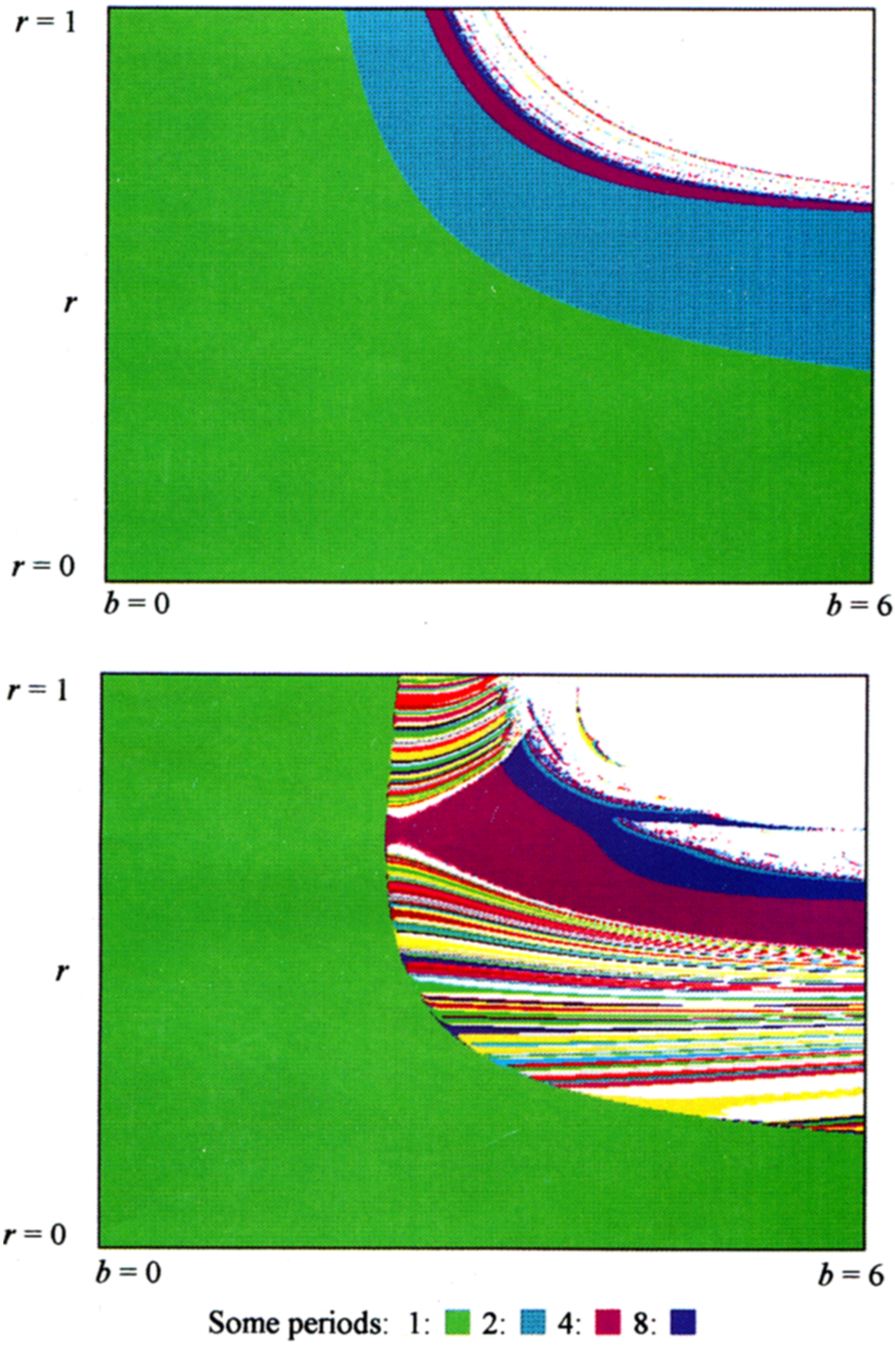


Fig. 10. (a) $\tau = 0$, $q = 1.2$, $s = 1.2$, $0 \leq r \leq 1$, $0 \leq b \leq 6$; (b) $\tau = 1$, $q = 1.2$, $s = 1.2$, $0 \leq r \leq 1$, $0 \leq b \leq 6$. For more detailed data for the periods, see the Appendix.

$$F\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} f_1(u_t, v_t) \\ f_2(u_t, v_t) \end{pmatrix}, \tag{25}$$

where

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} g_t \\ g_{t-1} \end{pmatrix} = x_t$$

and $F = x_{t+1}$. The derivative function dF has complex eigenvalues λ and $\bar{\lambda}$. The equation $|\lambda| \leq 1$ gives the stability condition

$$r \leq -\frac{1}{\varepsilon}. \tag{26}$$

The interpretation of this condition is the same as in the case of zero time lag (24). The notable difference is in the unstable situation. Chaos is not coming through period-doubling bifurcations but through Hopf bifurcation. The fixed point becomes repelling.

3.2. *The nonlinear market effect*

Now the system is

$$\begin{cases} g_{t+1} = g_t - r g_t^q + a(g_{t-\tau}) \\ a(g) = bg\, e^{(1/s)(1-(g/m)^s)}. \end{cases} \tag{27}$$

In the case of zero time delay the stability condition for the limit point can be presented in the form

$$r < \frac{2}{(q - \varepsilon)(m(1 - \varepsilon)^{1/s})^{q-1}}, \tag{28}$$

where parameter $q > 1$ and the goodwill elasticity of advertising at the target is

$$\varepsilon = 1 - \left(\frac{g^*}{m}\right)^s. \tag{29}$$

Equation (28) means that, when the depreciation rate of goodwill is high, the goodwill elasticity of advertising at the target must be low in a stable situation. The interpretation is the same as in the linear case (see Fig. 5). When parameter r is low, a firm can be very elastic in relation to advertising.

When the time delay $\tau = 1$ the stability condition for a fixed point attractor is

$$r \leq -\frac{(g^*)^{1-q}}{\varepsilon}. \tag{30}$$

The result is qualitatively the same as the condition in the linear case. The stable region becomes smaller when the exponent q increases.

Figure 9 presents an example of the bifurcation diagrams with fixed parameters b, s and q, so that immediate and lagged effect of advertising (cases (a) and (b), respectively) on the behavior of goodwill can be seen with different values of parameter r.

Figure 10(a) and (b), which has been drawn using the period testing algorithm, shows that the stable regions of the nonlinear model are in both cases ($\tau = 0$ and $\tau = 1$) larger than in the linear case. Figure 9(a) and (b) again describes special cases of Figure 10(a) and (b), respectively: the advertising expenditure parameter b is fixed at 3 (in both cases the parameters $s = q = 1.2$ are fixed). In Fig. A2 of the Appendix the periods of Fig. 10(b) have been drawn up to seventy periods and separated into five subpictures.

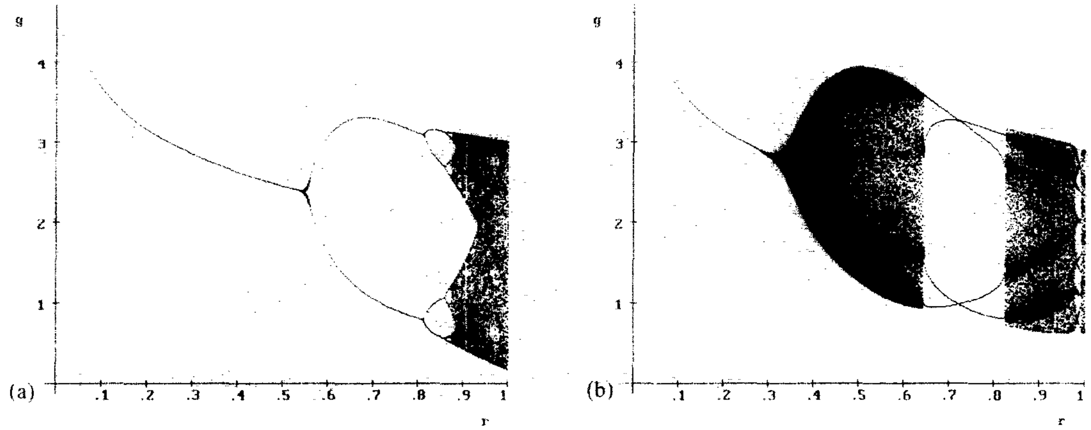


Fig. 9. The values of goodwill with different values of parameter r when $b = 3$, $s = 1.2$, $q = 1.2$, $\tau = 0$ and $\tau = 1$.

3.3. *The bounded case*

In the bounded case the system is

$$\begin{cases} g_{t+1} = g_t - rg_t + a(g_{t-\tau})\left(1 - \dfrac{g_t}{G}\right) \\ a(g) = bg\,e^{(1/s)(1-(g/m)^s)} \end{cases} \tag{31}$$

and the equilibrium goodwill g^* is

$$g^* = G\left(1 - \left(\frac{r}{b}\right)e^{-\varepsilon/s}\right). \tag{32}$$

The stability condition for the limit point in the case of zero time delay is now

$$r < \frac{2}{\dfrac{G}{G - g^*} - \varepsilon}, \tag{33}$$

which approaches the stability condition of the linear case, when $G \to \infty$. The behavior of goodwill is quite stable as in the previous cases when the value of the target goodwill g^* is not near the value of the saturation goodwill G (see Fig. 11(a)). Problems arise if a firm sets its target goodwill near the saturation level (see Fig. 11(b)). The situation is very unstable unless the depreciation rate of goodwill is quite low.

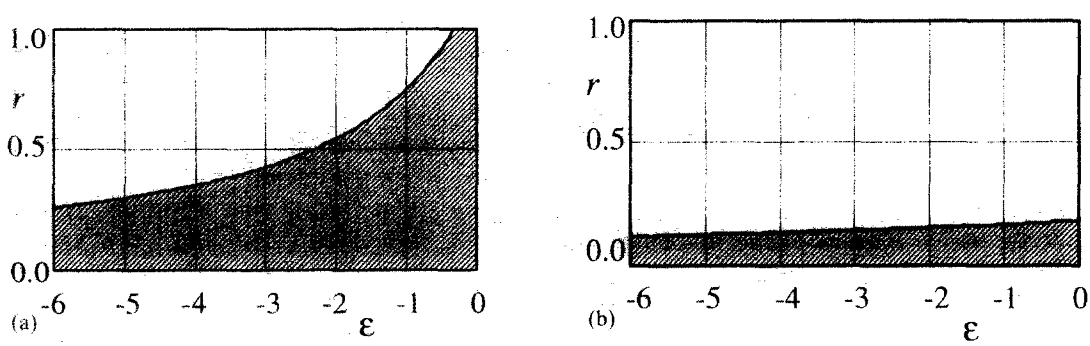


Fig. 11. Values for parameters r and ε, when the system has a fixed point attractor ($g^* = 2$, $G = 5$ and $G = 2.2$).

The stability condition for the fixed point attractor in the case, when the time lag $\tau = 1$, is

$$r \leqslant -\frac{1 - \frac{g^*}{G}}{\varepsilon}. \tag{34}$$

This condition has its limit in the condition, eqn (24) of the linear case, when $G \to \infty$.

When the depreciation rate of goodwill r is low, the situation is usually stable in all the three cases (linear, nonlinear and bounded) developed in this study. When parameter r is increasing, the situation becomes more complicated. Because the depreciation rate of goodwill r and the upper boundary of goodwill G are factors which depend on the environment, it is difficult for a firm to influence these factors. The goodwill elasticity of advertising at the target ε is a factor which can be changed within the bounds of a firm's possibilities in relation to its target goodwill g^* and the amount of money b. The goodwill elasticity of advertising at the target has a close relationship with parameter s of the model. This parameter together with the depreciation rate of goodwill are the most important variables in this model leading either to a stable or an unstable situation in the behavior of goodwill.

The bifurcation diagrams (see Fig. 12) illustrate the cases of zero time delay in which the upper limit G and the advertising expenditure b are fixed. The difference between the diagrams is that in Fig. 12(a) parameter s (linked with advertising policy) is fixed and parameter r is varying. In Fig. 12(b) the depreciation rate r is fixed and parameter s is varying.

Next the bifurcation diagrams are drawn, when the time lag is three ($\tau = 3$). In the first figure (Fig. 13(a)) parameter r is varying and in the second figure (Fig. 13(b)) parameter s is varying.

Notable in the bifurcation diagrams (Figs 12 and 13) is that also parameter s is a very important factor leading the system to chaos. Parameter s is in a close relationship with the goodwill elasticity of advertising. It is seen that when the firm is too elastic chaos may appear in the behavior of goodwill. If a system with time delay is concerned the behavior

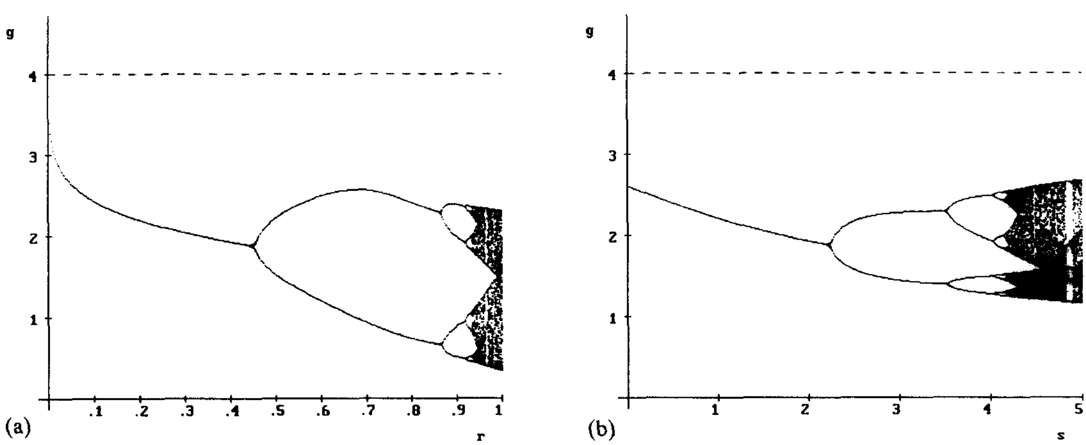
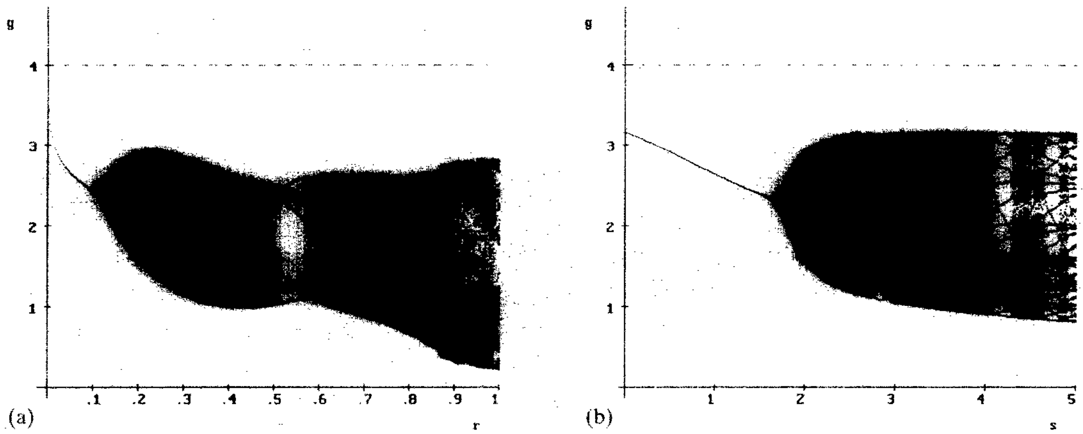


Fig. 12. The values of goodwill with different values of (a) parameter r, when $G = 4$, $b = 3$, $s = 2$; (b) parameter s, when $r = 0.4$.



(a) (b)

Fig. 13. The values of goodwill with different values of (a) parameter r, when $G = 4$, $b = 3$, $s = 2$; (b) parameter s, when $r = 0.2$.

of goodwill becomes easily unstable if advertising is finished abruptly at the target. This is the case when the parameter $s > 1.8$ (see Fig. 13(b)) in the advertising function (2.12).

Figure 14(a) and (b) are generalizations of the cases presented in Figs 12(a) and 13(a), respectively. They have been drawn varying both of the parameters b and r. In Fig. 14(b) ($\tau = 3$), a new type of stable fixed point region can be obtained. The case $\tau = 0$ (Fig. 14(a)) is almost identical to the linear case (Fig. 7(a)).

Generally, the bounded model doesn't differ essentially from the linear model, if the target goodwill is clearly lower than the saturation level of goodwill. The model can be used in the case when the target goodwill is near the saturation level and goodwill is varying inconveniently. By using the model it can be studied how much the elasticity or target goodwill must be changed to get out of this inconvenient situation.

4. SUMMARY

Starting from the classical advertising capital model, the Nerlove–Arrow model, a time delayed feedback model between advertising and goodwill was developed. It was assumed that the effects of advertising on the equilibrium goodwill can be linear, nonlinear or bounded. When studying these cases in detail, each one of the versions of the model worked well, i.e. each kind of goodwill behavior could be observed in all versions and interpretations of the stability conditions for the fixed point attractors were reasonable.

On the whole, a firm may face serious problems when the depreciation rate of goodwill is too high. Advertising is important in this case but mistakes in advertising are fatal. In the monopolistic case, as assumed in this study, goodwill would be estimated best with the help of sales. If a firm reacts in a wrong way when attempting to estimate whether the target goodwill has already been reached, periodic or chaotic behavior of goodwill may follow. When a firm is too elastic in decreasing its advertising or the time delay of advertising is too long, the firm will lose its control of the customers' behavior.

The ideas in this study form a good basis for further research. The continuous advertising function and the three different kinds of effects of advertising could be the basis when attempting to apply the model to a duopolistic case. Another approach could be to try to find out the possibilities of using the integro-differential equation mode.

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Figs 14 and A1–A3 appear on pp. 2101–2104.

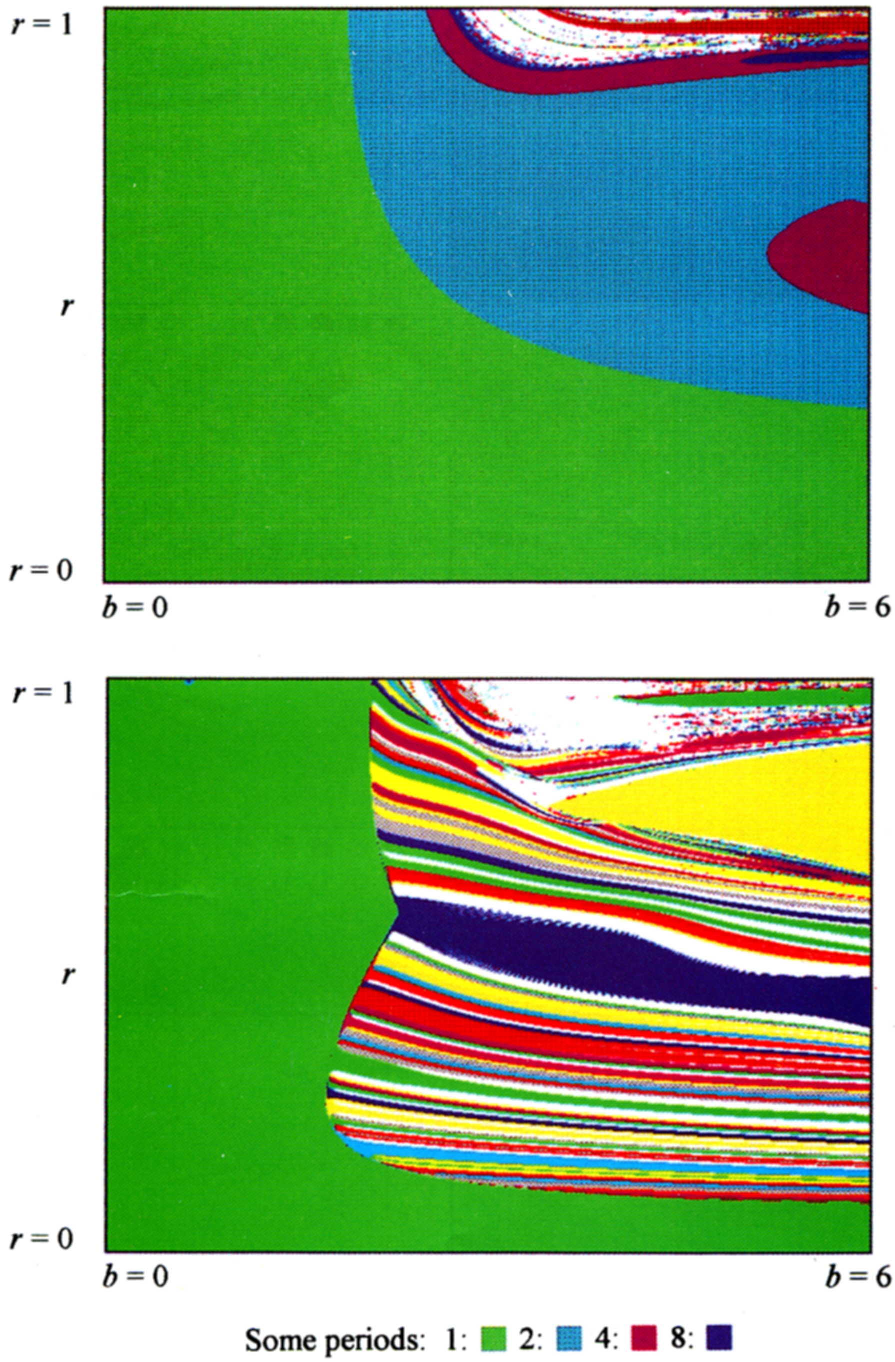

Fig. 14. (a) $\tau = 0$, $G = 4$, $s = 2$, $0 \leq r \leq 1$, $0 \leq b \leq 6$; (b) $\tau = 3$, $G = 4$, $s = 2$, $0 \leq r \leq 1$, $0 \leq b \leq 6$. For more detailed data for the periods, see the Appendix.

APPENDIX

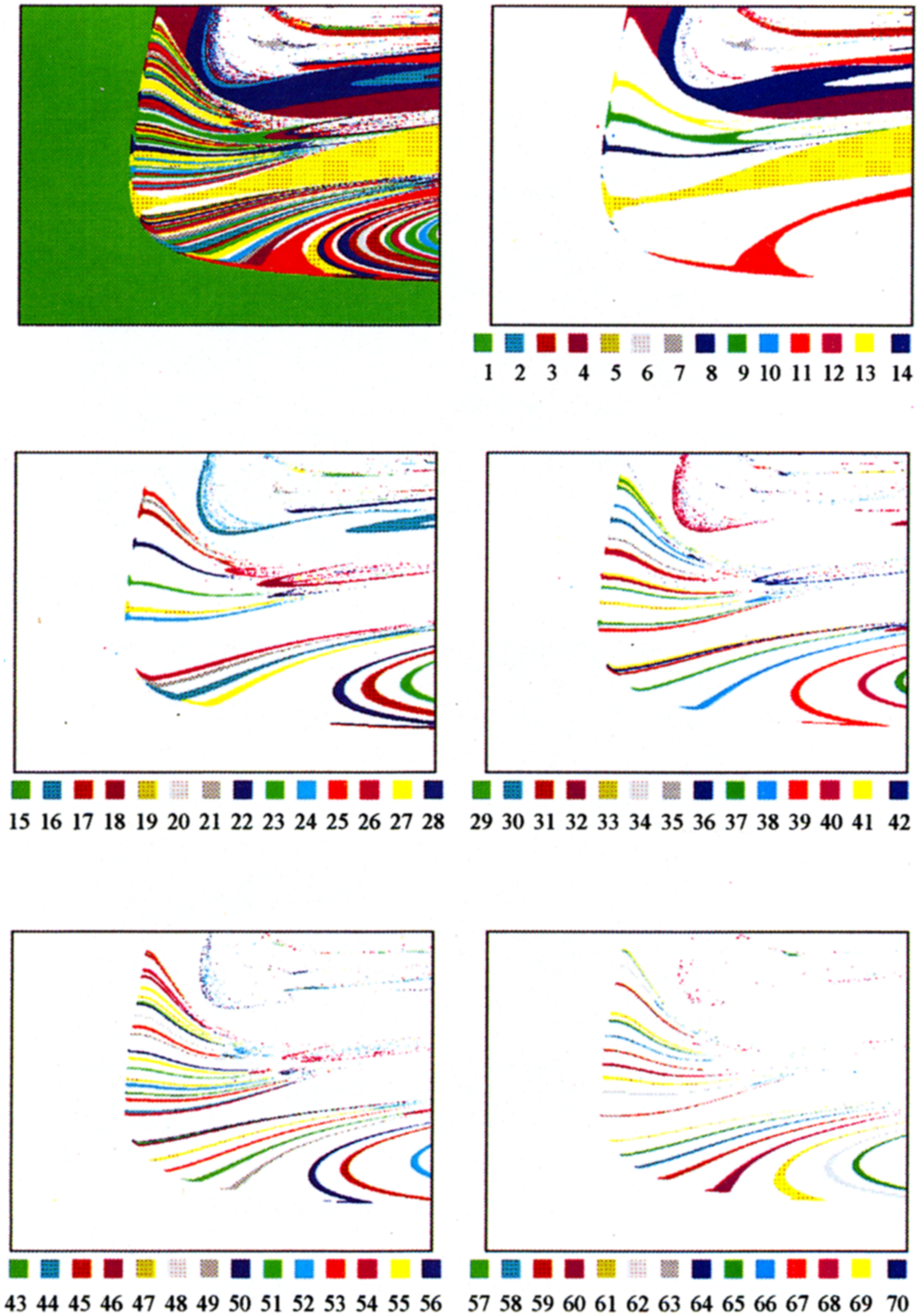

Fig. A1. Linear model $\tau = 1$, $s = 2$, $0 \leqslant r \leqslant 1$, $0 \leqslant b \leqslant 5$ (see Fig. 7(b)).

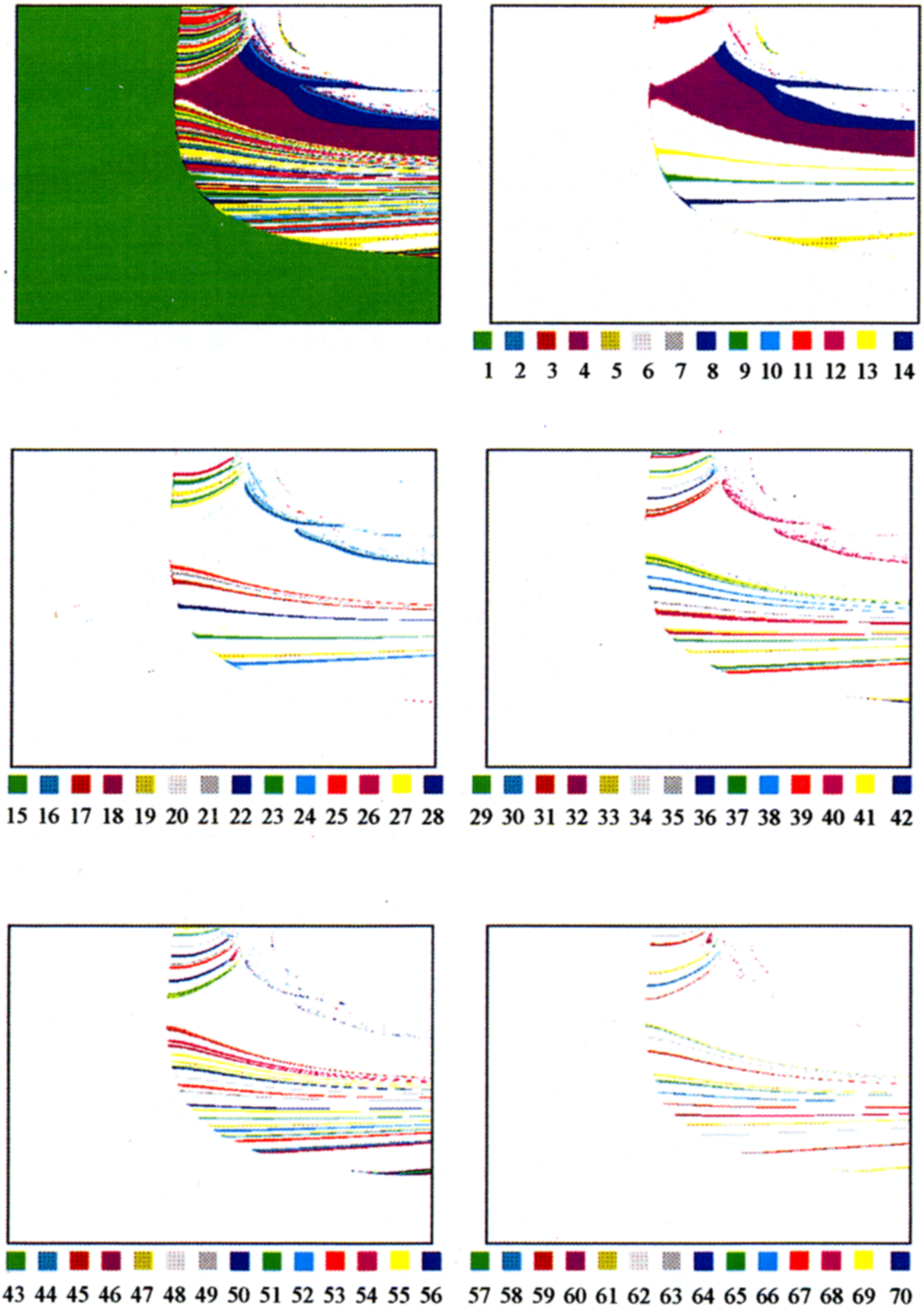

Fig. A2. Non-linear model $\tau = 1$, $q = 1.2$, $s = 1.2$, $0 \leqslant r \leqslant 1$, $0 \leqslant b \leqslant 6$ (see Fig. 10(b)).

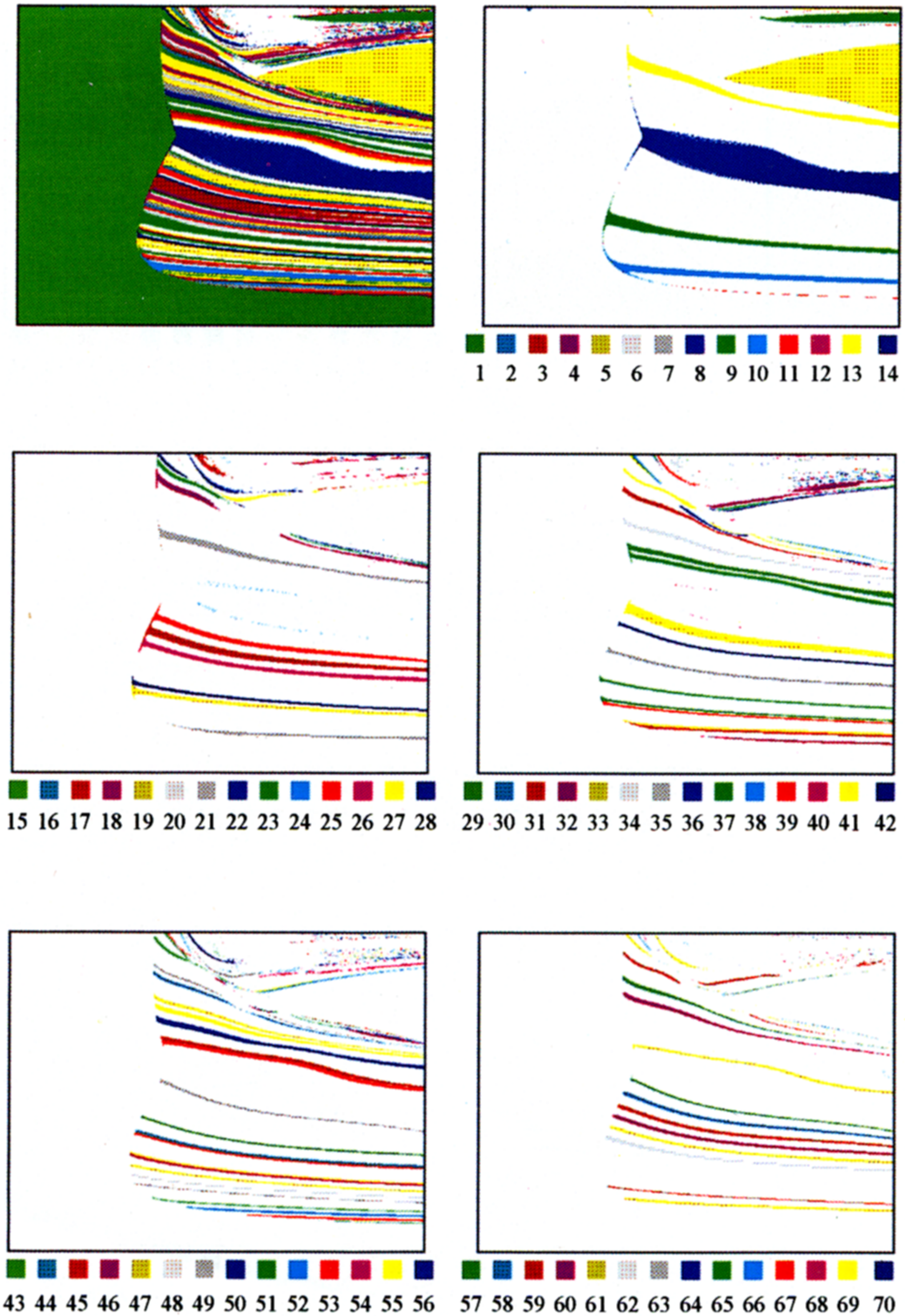

Fig. A3. Bounded model $\tau = 3$, $G = 4$, $s = 2$, $0 \leq r \leq 1$, $0 \leq b \leq 6$ (see Fig. 14(b)).