The Art and Science of Decision-Making

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ANALYZING THE BEHAVIOR OF A NON-LINEAR ADVERTISING CAPITAL MODEL; AN APPLICATION OF BIFURCATION THEORY, LYAPUNOV EXPONENTS AND CORRELATION DIMENSION

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ÅBO 1996

ÅBO AKADEMIS FÖRLAG - ÅBO AKADEMI UNIVERSITY PRESS

ANALYZING THE BEHAVIOR OF A NON-LINEAR ADVERTISING CAPITAL MODEL; AN APPLICATION OF BIFURCATION THEORY, LYAPUNOV EXPONENTS AND CORRELATION DIMENSION

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Abstract

In this study a time delayed feedback model describing the relations between advertising and goodwill is introduced. The model has its origin in the classical Nerlove-Arrow advertising capital model (Nerlove and Arrow 1962). A continuous advertising function is used and the non-linear market effect of advertising on the dynamics of goodwill is employed. In the case of a lagged effect of advertising the dimension of the model exceeds unity. This means an unstable situation in the limiting behavior, i.e. in the bifurcation point an invariant closed curve is born as the attracting fixed point becomes repelling. After this standard Hopf bifurcation the behavior of goodwill is periodic or nearly periodic untill subsequent period doubling bifurcations may happen leading the system to chaos with increasing parameter values. This period doubling route to chaos is analyzed numerically by bifurcation diagrams and by the techniques of the Lyapunov exponents and the correlation dimension. The stability conditions for the fixed point of the model is determined analytically.

1. Introduction

The aim of the present work is to study the suitability of the chaos theory and its techniques for clarifying the temporal behavior of an advertising model used in marketing studies. The dynamic advertising model used in this paper was introduced by Luhta (1994) and is based on the classical Nerlove-Arrow model (Nerlove and Arrow, 1962). The general purpose is to model and study the behavior of the goodwill of a firm. Goodwill is defined in accordance with the Nerlove-Arrow model. The techniques of the calculus of variations and the optimal control theory have numerous applications in the area of marketing and of advertising especially (see e.g. Sethi, 1977; Sethi and Thompson, 1981). These models are all optimization models. In the state space approach of the control theory the purpose is to find an optimal control (see e.g. Virtanen, 1982), i.e. in the case of the Nerlove-Arrow model to solve an optimal advertising policy so that the optimum level of goodwill of a monopolistic firm could be reached. Another approach to study the non-linear economic relations is based on the analysis of empirical time series (see e.g. Booth et al, 1994; Hibbert and Wilkinson, 1994).

The techniques for analyzing non-linear dynamical systems are far less developed than the techniques for linear models. In detail, it is usually not possible to solve a non-linear dynamical system analytically. What is left to an analysis of non-linear systems is the description of the qualitative behavior of the system. Therefore, the bifurcation theory as a part of the chaos theory is a subject which is becoming more and more important in economic dynamics. Central to this topic is the question whether the qualitative properties of a dynamical system change when one parameter is changing (Lorenz, 1993). The bifurcation behavior of the system depends to some degree on the time concept involved and is different with one-dimensional and higher-dimensional cases.

In this study a time delayed feedback model between advertising and goodwill is developed using a continuous non-linear advertising function. The behavior of the system is formulated in discrete time. The lagged effect of advertising, i.e. the market behavior is assumed to be non-linear. The behavior of the model is analyzed in more detail in the two-dimensional case, i.e. when time lag $\tau=1$. The stability conditions for fixed point attractors, i.e. for a stable limiting behavior of goodwill are defined both analytically and numerically. Numerical methods including bifurcation diagrams and Lyapunov exsponents are used to clarify periodic, nearly periodic and chaotic nature of the behavior of goodwill with varying parameters. If the time lag τ exceeds one the behavior of goodwill seems to be qualitatively the same as in the two-dimensional case. There is no theory to prove it mathematically (Lorenz, 1993) but using the correlation dimension technique the similarity can be shown.

The paper is organized as follows. Section 2 presents the classical Nerlove-Arrow model. Then the ideas of linear, non-linear and bounded effects of advertising are introduced. The market behavior is derived using first the assumption of constant advertising. Then advertising is allowed to vary. The advertising policy of the management is incorporated into the model via an advertising function. Then the non-linear model with lagged advertising effect and the stability condition of equilibrium are derived. Section 3 presents the basic concepts of the chaos theory used in this paper, and the model is analyzed using both analytical and numerical methods. The paper ends with summary and conclusions in Section 4.

2. The Non-linear Advertising Capital Model

2.1. The Nerlove-Arrow Model

Advertising expenditure is in many ways similar to investments in durable plant and equipment. The latter affects the present and future net revenue of the investing firm. Advertising expenditures on their part affect the present and future demand for the product and, hence, the present and future net revenue of the firm's advertising. It has been plausible demonstrated, that the necessary conditions for a maximum of the present value of future net revenues lead to a decision rule which is similar to a rule of thumb which is actually used in many firms (Nerlove and Arrow, 1962).

The demand for the output of an individual firm or of an industry depends on advertising expenditure in addition to the price of the product and consumer incomes as well as to the prices of competing or complementary products. Advertising expenditure may shift the demand function with new customers, some of whom may never have consumed the product before in the case of an industry, and some of whom have previously consumed the product of another firm in the case of an individual firm (Nerlove and Arrow, 1962).

Despite its precise effects on the demand function, advertising expenditure, at any one time, may be assumed to lose its effectiveness in subsequent periods. An advertising campaign in progress may bring a hundred thousand customers into the fold today, but next month or next year many of these will have drifted off. Furthermore, it is difficult to bring about permanent changes in consumer tastes and preferences because there is a tendency for the preferences of consumers to return to their old pattern. On the other hand, the effects of a given advertising campaign, in relation to the number of consumers and their tastes, tend to diminish steadily due to the persistance of old patterns for a considerable period following the campaign, albeit for the reason given (Nerlove and Arrow, 1962).

One way to present the temporal differences of the effects of advertising on the demand would be to include a number of past advertising outlays in the demand function. However, such an approach is not especially useful. A more promising analytical approach, and one which has considerable intuitive appeal, is defining a stock which is called goodwill and noted by $g(t)$. Goodwill is supposed to summarize the effects of current and past advertising outlays on demand. If the price of a unit of goodwill is \$ 1, then a dollar spent on advertising increases goodwill by an equal amount. On the other hand, a dollar spent some time ago should, according to the previous argument, contribute less. One possible way of presenting this lesser contribution is to say that goodwill, like many other capital goods, depreciates. If it is further assumed that current advertising expenditure cannot be negative and that depreciation occurs at a constant proportional rate, r, we get the equation

$$\frac{dg(t)}{dt} + r \cdot g(t) = a(t) \geq 0, \tag{2.1}$$

where $a(t)$ is current advertising outlay and $a(t)$ and $g(t)$ are functions of time. Equation (2.1) states that the net investment in goodwill is the difference between the gross investment (current advertising outlay) and the depreciation of the stock of goodwill (Nerlove and Arrow, 1962).

2.2. Market Reactions to Advertising

The Nerlove-Arrow Equation (2.1) assumes that there is no time lag between advertising expenditures and increases in the stock of goodwill. As the demand for the monopolist's product is a function of the stock of goodwill, that implies that the rate of sales at time t totally adjusts itself immediately to the rate of advertising prevailing at time t (Pauwels, 1977). Introducing a time lag between the rate of advertising and its effect on the rate of sales leads to a control problem in which the equation of motion is given by an integro-differential equation. There are many generalized Nerlove-Arrow models dealing with distributed time lags in this way (e.g. Bensoussan et al, 1973; Hartl, 1984; Mann, 1975; Pauwels, 1977).

If it is assumed that the change of goodwill at time t depends on the amount of advertising in a period a certain time ago, a fixed time delay τ can be incorporated into Equation (2.1) yielding

$$\frac{dg(t)}{dt} = a(t-\tau) - r \cdot g(t). \tag{2.2}$$

In the earlier models with time delays it had been assumed that advertising was independent of sales. When the advertising budget was, however, assumed to depend on sales, which, on the other hand, depended on the goodwill stock, the amount of advertising could be described as a function of goodwill (Luhta, 1994; Luhta and Virtanen 1996). Choosing different values for a in (2.1) or (2.2), many different advertising policies could be incorporated into the model.

In the classical Nerlove-Arrow model it is assumed that a dollar spent on advertising increases goodwill by an equal amount of money. It is assumed, in other words , that the marginal cost of goodwill is constant. This linear relation can be motivated, for example, via the model (2.1) as follows.

Let us assume, for the moment, that advertising is held constant all the time. Take the discrete formulation of the model (2.1):

$$g_{t+1} = g_t - r \cdot g_t + a \tag{2.3}$$

and consider the behavior of the system towards the equilibrium. It is easy to see that the equilibrium state is stable and the equilibrium level of goodwill is

$$g^* = \frac{a}{r}. \tag{2.4}$$

This means that the equilibrium goodwill g^* is linearly related to advertising as assumed in Nerlove-Arrow model. The marginal cost of goodwill is, therefore, constant in equilibrium. It can be further seen that the equilibrium goodwill is inversely proportional to the depreciation rate r of goodwill.

The linear relation does not seem very reasonable in practice wherefore it is more reasonable to assume that the marginal cost of goodwill is increasing. This can be motivated, for example, with the forgetting effect of the system. The increasing marginal cost of goodwill can be incorporated into the model by adding an exponent $q > 1$ to the depreciation term

(2.5) $$g_{t+1} = g_t - r \cdot g_t{}^q + a.$$

This means nonlinearity of the model at the same time. The equilibrium goodwill is now

(2.6) $$g^* = \left(\frac{a}{r}\right)^{\frac{1}{q}}.$$

Another natural approach is to assume that there is a saturation level for the equilibrium goodwill. This can be modelled by adding an upper bound G for the goodwill in model (2.3). The transient dynamics of the system is then

(2.7) $$g_{t+1} = g_t - r \cdot g_t + a \cdot (1 - \frac{g_t}{G}).$$

The equilibrium goodwill from model (2.7) is

(2.8) $$g^* = \frac{aG}{rG + a}.$$

Equation (2.8) also means increasing marginal cost for goodwill.

It has been assumed above that the advertising outlay is constant along time. The assumption was made only for finding a way to model the effect of markets on the development of goodwill. Three different effects were considered: the linear effect (2.3) and (2.4), the non-linear effect (2.5) and (2.6), and the bounded case (2.7) and (2.8). Figure 1 depicts the three market effects from the point of view of marginal cost of goodwill, i.e. presenting advertising outlay a as a function of goodwill in the equilibrium. In the following, we give up the assumption of constant advertising and consider the system under non-constant advertising environment via different advertising policies.

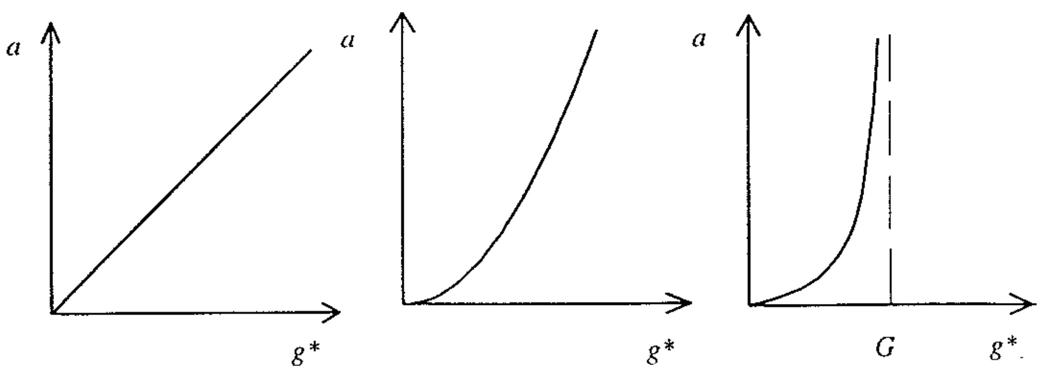

Figure 1:
Linear marginal cost (linear effect) | Non-linear marginal cost (non-linear effect) | Non-linear marginal cost (bounded effect)

2.3. Advertising Policies

It is assumed that the advertising policy of the firm, i.e. the advertising function obeys a law of the following type (Luhta, 1994):

$$a(g) = b \cdot g \cdot e^{\frac{1}{s}\left(1-\left(\frac{g}{m}\right)^{s}\right)}. \tag{2.9}$$

Figure 2 presents a scematic description of the model. For low values of goodwill g, the management can increase goodwill with increasing advertising even to the maximum possible amount of money $(=\bar{a})$ allowed by the chosen policy. The larger the value of the scale parameter b is the larger is the advertising budget and the value of $\bar{a}$. Further, a target goodwill is assumed to exist and is denoted by g^*. The corresponding advertising expenditure is a^*. (The target goodwill is denoted by g^* because the analysis will be concentrated in the following on equilibrium target goodwills only).

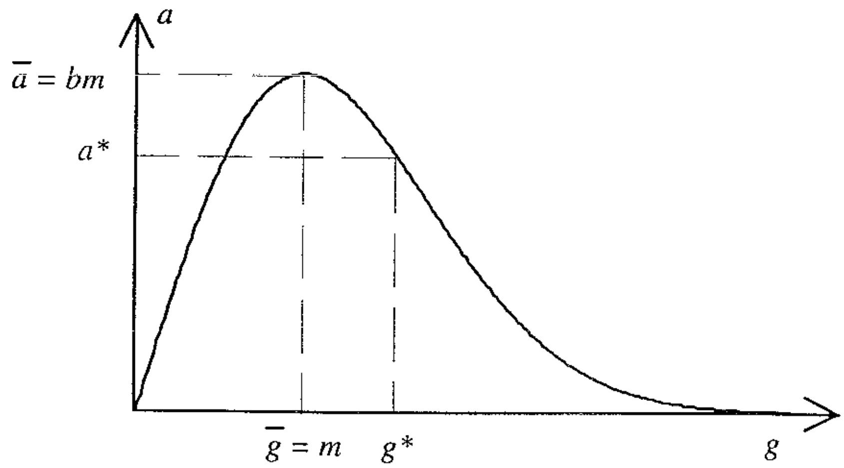

Figure 2: Advertising policy: dependence of advertising outlay on goodwill

It is natural to assume that in the beginning goodwill is smaller than the target goodwill, $g_0 < g^*$. Therefore, an increasing amount of money is first spent on advertising, and before the target value of goodwill will be reached, the highest amount of money $(= \bar{a} = bm)$ allowed by the budget may have to be spent on advertising (at the level $\bar{g} = m$ of the goodwill). After the advertising outlay has reached its peak it starts to decrease. It is natural to assume that investment in advertising will not be ceased abruptly but, on the contrary, the firm continues to advertise to get a better and better image. The exponent s describes the ability of the firm to maintain its goodwill with the advertising policy chosen.

Function (2.9) can be regarded as a reaction function for the estimated goodwill. When the parameter s is large the advertising flow $a(g)$ decreases quickly after the value m whereas with small values of s the decrease in advertising outlay is slower. Setting the value for the parameter s is a long-term decision in the firm. Therefore, it is important that the management defines the advertising policy of the firm properly when attempting to attain more permanent goodwill for the firm.

In Figure 3 a non-linear market effect is assumed to exist (the curve $a^* = r \cdot (g^*)^q$). In the case of a stable equilibrium, the target goodwill g^* is maintained with the advertising outlay a^*. Advertising policies a_1 to a_4 represent the family of different advertising policies to reach the target goodwill g^*. The policy to be used is chosen by fixing the values for the parameters

b, m, and s. If there exists a fixed upper bound for the advertising budget (a_{max} in the figure), some of the policies may be infeasible (a_4 in the figure) (Luhta and Virtanen, 1996).

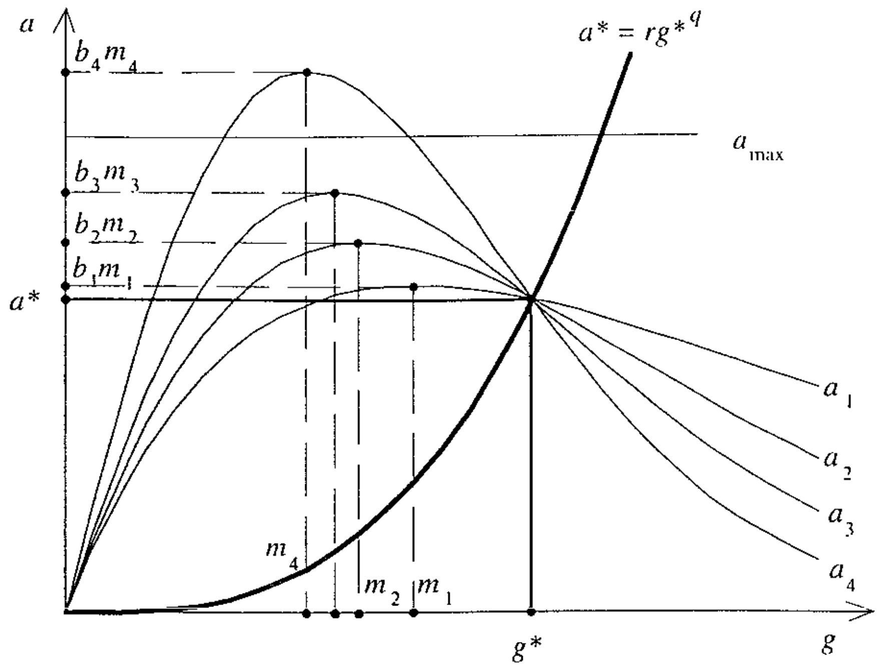

Figure 3: The family of different advertising policies to reach the stable target value g^* for goodwill (non-linear market effect)

2.4. The Non-linear Model with Lagged Advertising Effect

Depending on the assumed market behavior, three different versions of the model, i.e. the linear, non-linear and bounded effect models, are obtained. The reaction function for the goodwill is Equation (2.9). The reaction of advertising is either immediate or time delayed. The version studied in this paper is the system with the lagged non-linear market effect. The model thus becomes

$$(2.10) \qquad \begin{cases} g_{t+1} = g_t - r \cdot g_t^q + a(g_{t-\tau}) \\ a(g) = b \cdot g \cdot e^{\frac{1}{s}\left(1-\left(\frac{g}{m}\right)^s\right)} \end{cases} .$$

The analysis of the model concentrates especially on the derivation of the stability condition for the fixed point of the system. When advertising is non-constant, the stability of equilibrium depends on the local properties of the advertising function, which, on the other hand, is a function of goodwill. These local properties can be described in terms of the goodwill elasticity of advertising at the target

$$(2.11) \qquad \varepsilon = \frac{g^*}{a^*} \cdot \frac{da}{dg}\bigg|_{g=g^*} = \frac{g^* \cdot a'(g^*)}{a(g^*)} = 1 - \left(\frac{g^*}{m}\right)^s .$$

The equilibrium goodwill g^* is supposed to be the target. In economics it is common to use elasticity when describing the properties of the system in the neighbourhood of equilibrium. In the theory of dynamic systems, the same local properties can be studied by means of the following stability condition (Devaney, 1986; Feichtinger, 1992). In the case τ=1 the function F is used:

$$(2.12)\qquad F\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} f_1(u_t, v_t) \\ f_2(u_t, v_t) \end{pmatrix},$$

where $\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} g_t \\ g_{t-1} \end{pmatrix} = x_t$ and $F = x_{t+1}$.

The relation $|\lambda| \le 1$ gives the stability condition, where λ and $\bar{\lambda}$ are the complex eigenvalues of the derivative function dF.

3. Analysis of the Model

3.1. Basic Concepts of Chaos theory

The set of all possible states of the system is called the state space. Changes in the state of the system can be presented in the form of a curve in the state space. Each point of this curve carries a label which records the time of the observation. This is called an orbit of the system. The history of a system is presented graphically as an orbit in a geometric state space. The state space, which is filled with orbits, is called the phase portrait of the dynamic system. The orbits of the phase portrait represent the behavior of the system which is just being modelled. Orbits occur in the form of limit sets. When studying the qualitative predictions of asymptotic behavior, the phase portrait for asymptotic limit sets must be examined (Abraham and Shaw, 1984).

When using the dynamic system to model the experimental situation, an orbit will model the start-up transient, while its limit set models the equilibrium state which follows. The asymptotic approach of the orbit to its limit set models the dying away of the transient as the system settles to its dynamic equilibrium. The only equilibrium states which may be observed experimentally are those modelled by the limit sets which receive most of the orbits. These are called attractors. An attractor is a limit set with an open inset. That is, there is an open neighbourhood of the limit set within its inset. The inset of an attractor is called its basin (Abraham and Shaw, 1984; Devaney, 1986).

A limit set can be a fixed point. If this fixed point is stable, it is called a fixed point attractor. If it is unstable, a dynamic system may tend, especially in two-dimensional state spaces, to limit cycles which are either periodic or nearly periodic attractors or repellors. In higher dimensions unusual limit sets are called chaotic attractors. The behavior of an orbit of a fully deterministic dynamic system attracts then, in the long run, to an attractor which is totally unpredictable. Any small error in the measurement of the current state eventually leads to total ignorance of the position of the orbit within the chaotic attractor. The exact position of the orbit at a given moment is unknown. Small differences in its current position leads later to an enormous difference in its position (sensitive dependence on initial conditions). The orbit will eventually come arbitrarily close to any point on the thick bands (topological transitivity) (Abraham and Shaw, 1984; Medio, 1992; Seydel, 1994).

The illustration of the attractors of the system can be done with the help of bifurcation diagrams. Central to this topic is the question whether the qualitative properties of a dynamical system change when one parameter is changing. Parameters are introduced into an economic model in order to reflect the influence of exogenous forces which are either beyond the scope of pure economic explanation or which are intentionally considered as being exogenously given from the point of view of partial theorizing. It is desirable to determine whether the qualitative behavior of a dynamical system persists under variations in the parameter space. The bifurcation phenomena can be related to the notion of structural stability. Roughly speaking, a dynamical system is called structurally stable if the qualitative dynamic properties of the system persist with small variations in its structure, i.e., when varying the parameters or considering small changes in the functional forms. In other words, a dynamical system is structually stable if the two orbits stay close together. A bifurcation value is therefore a value for which the dynamical system is structually unstable (Lorenz, 1993).

There are two basic types of bifurcations, i.e. in one-dimensional systems bifurcations are often period doubling bifurcations and in higher-dimensional systems Hopf bifurcations. When a bifurcation is of the period doubling type it means that at the bifurcation point where the fixed point becomes unstable it divides into two, splits apart. With a further increasing parameter value, chaos becomes after these period doubling bifurcations. This behavior is called a period doubling route to chaos. A Hopf bifurcation happens at least in two-dimensional systems at the moment when the fixed point becomes unstable, i.e. in the bifurcation point an invariant closed curve is born as the attracting fixed point becomes repelling. The order of the attractor is not easily seen from the bifurcation diagram. An efficient measure in assorting different types of attractors, especially chaotic attractors is the Lyapunov exponent. Drawing Lyapunov exponents together with bifurcation diagrams the periodic, the nearly periodic and the chaotic behavior of the model can be determined.

Two key aspects of chaos are the stretching of infinitesimal displacements and the existence of complex orbit structure in the form of a vast variety of possible orbits. The stretching property is closely related to sensitive dependence on initial conditions. The existence of complex orbit structure is exemplified in the exponential increase, as a function of itinerary lenght, of the number of distinct symbol sequences that represent orbits. It is important to have quantitative ways of characterizing these two aspects (Ott et al, 1994). A quantitative characterization of streching properties is provided by the Lyapunov exponent, while a characterization of complex orbit structure is provided by entropy.

Alternative definitions emphasize sensitivity to initial state. If *V* is a little state-element round the point of an attractor and *F(V)* is its image in the mapping (2.12) then the state-element flattens towards the attractor according to the property of the attractor. Because the system is sensitive to initial conditions, the state-element must strech in the direction of the attractor. An average exponential rate of the divergence can be measured by using the Lyapunov exponent. The exponents can be computed locally from the eigenvalues of the derivate mapping of *F* (Wolf et al, 1985). The system is sensitive to the initial condition if the average exponential rate of the divergence is positive.

The system of the present study can be specified by the map $g_{t+1} = F(g_t)$. If g^0 is a point near the point g_t the absolute value of the distance of the maps $F(g_t)$ and $F(g^0)$ is $\left|F(g_t) - F(g^0)\right| \approx \left|F'(g_t)\right| \cdot \left|g_t - g^0\right|$. The Lyapunov exponents are calculated so that they are the mean values of the logarithms of the largest eigenvalues of the derivative maps $F'(g_t)$ at each point g_t. So Lyapunov exponents are calculated along the attractor of the system. The calculations are done in the two dimensional case which means that there are two Lyapunov exponents. The Lyapunov exponents can be interpreted as follows. When all Lyapunov exponents are negative on an attractor, the attractor is periodic. When the largest Lyapunov exponent is zero and the other is negative the attractor is nearly periodic. In the case of a positive Lyapunov exponent the system is chaotic.

If the dynamics of a system is chaotic then, on the basis of the topological transitivity and sensitivity for an initial state, any orbit long enough fills the attractor quite well. Consequently, the properties of the attractor can be measured from a long orbit. The most convinient way to recognize the attractor is to determine its dimension. There are several different definitions of the dimension (see Falconer, 1990; Simm et al, 1987). The Hausdorff dimension is the original definition of the dimension. Unfortunately, there is no fast computer algorithm for calculating the Hausdorff dimension.

The statistic which helps to obtain topological information about the underlying system is the correlation dimension. For truly random data, the correlation dimension monotonically increases with the dimension of the space within which these data are contained. This latter dimension is called the embedding dimension. In contrast, for chaotic data the correlation dimension remains small even when the embedding dimension increases. The correlation dimension measure indicates the complexity of the data (Booth et al, 1994).

An effective computer algorithm for determining the correlation dimension was developed by Grassberger and Procaccia (1983) and focuses on the way points representing sequences of values scattered in *m*-space. In *m*-space *m* denotes the lenght of a sequence of time states ($x_i(t), x_i(t-1), \ldots, x_i(t-m+1)$) at succeeding time periods. This is called the *m*-history of the observation $x_i(t)$. The sequence of the vectors ($x_i(t), x_i(t-1), \ldots, x_i(t-m+1)$) is shorter than the original time series and varies with the value of *m*. The number *m* is called the embedding dimension. Each *m*-history describes a point in a *m*-dimensional space and the sequence of points will therefore form a geometric object which is equivalent to the appropriate object generated by the true dynamical system (Takens, 1981). If a data is random, the points will be randomly scattered throughout *m*-space, but if it is chaotic, they will cluster more. The correlation dimension measures the way the *m*-space is filled by datapoints (Hibbert and Wilkinson, 1994; Lorenz, 1993).

If two points x_i and x_j in an *m*-space are on the attractor then it is said that these two points are spatially correlated if the Euclidian distance is less than a given radius r of an *m*-dimensional ball centered at one of the two points, i.e. if $|x_i - x_j| < r$. The spatial correlation between all points on the attractor for a given *r* is determined by counting the number of these pairs located in a ball around every point, i.e.

$$(3.1) \quad C(r,m) = \lim_{N\to\infty} \frac{1}{N^2} \cdot (\text{number of pairs } (i,j) \text{ such that } |x_i - x_j| < r),$$

where *N* is the length of the series of constructed *m*-histories (Lorenz, 1993).

C(r,m) is called a correlation integral in the *m*-space. Using the correlation integral *C(r,m)* the correlation dimension of the *m*-space can be calculated and it is

$$(3.2) \quad D(m) = \lim_{r\to 0} \frac{\ln C(r,m)}{\ln r}.$$

The usual technique is to make a log-log-plot of the correlation integral *C(r,m)* as a function of the parameter *r*. The slope of the curve is then the correlation dimension.

3.2. Analysis of the Model

When the time delay $\tau = 1$ the stability condition for a fixed point attractor can be presented in the form

$$(3.1) \qquad r \le -\frac{\left(g^*\right)^{1-q}}{\varepsilon}.$$

This means that when the depreciation rate of goodwill *r* is high, it is possible to reach the target g^*, i.e. a stable situation, by means of low absolute values of goodwill elasticity ε only (see Fig. 4). In the case of a low depreciation rate, stability sets no requirements for the elasticity. The stable region becomes smaller when the exponent *q*, i.e. the non-linearity, increases.

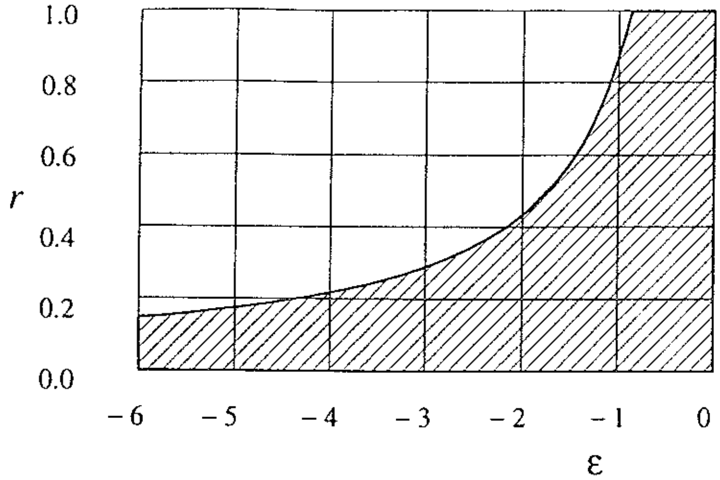

Figure 4: Values for parameters r and ε when the system has a fixed point attractor (g^*=2 and q=1.2): the shaded area

When the depreciation rate of goodwill r is low, the equilibrium state is usually stable in the model. When parameter r is increasing, the situation becomes more complicated. Because the depreaciation rate of goodwill r is a factor which depends on the environment, it is difficult for the firm to influence this factor. The goodwill elasticity ε of advertising at the target is a factor which can be changed only within the bounds determined by the target goodwill g^* and the amount of money b. The goodwill elasticity of advertising at the target has also a close relationship with parameter s of the model. The influence of parameter s can be studied with the help of the bifurcation diagram and the Lyapunov exponents. In Figure 5 the parameters have the values s=2, b=3, q=1.2 and the parameter r varies between 0 and 1.

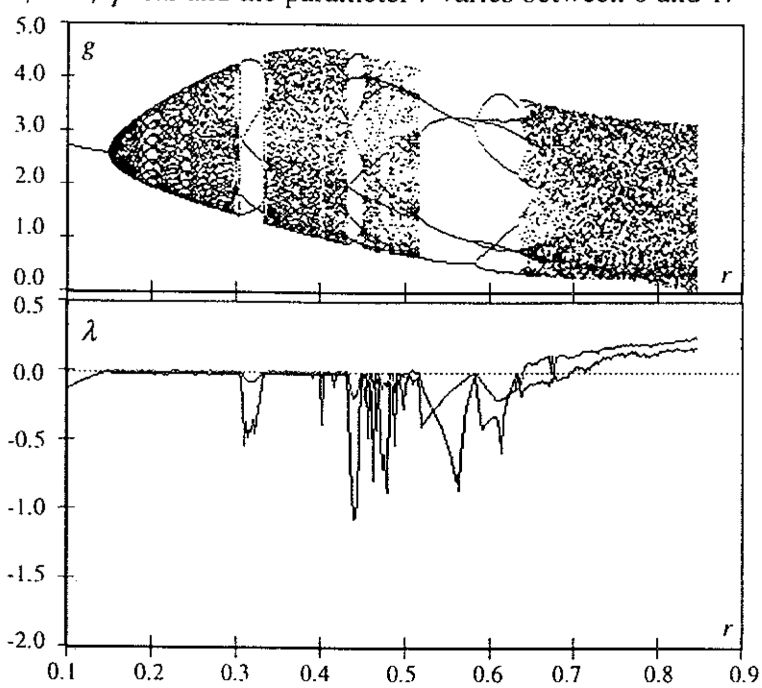

Figure 5: The behavior of goodwill with varying parameter $0.1<r<0.9$, when s=2, b=3 and q=1.2

Figure 5 shows that the largest Lyapunov exponent is zero for a long time after the Hopf bifurcation. This means that the behavior of the system must be nearly periodic. At the point where the depreciation rate of goodwill is $r \approx 0.30$ both Lyapunov exponents are negative and goodwill gets five different values. After that it is not easy to see either from the bifurcation diagram or from the Lyapunov exponents what really happens or what are the periods. For the parameter value $r \approx 0.52$ a 4-period begins and further for the value $r \approx 0.58$ a 8-period. Finally, with high values of parameter r chaos is evident because the largest Lyapunov exponent is positive. A more exact description of the behavior of goodwill can be obtained via zooming, i.e. by drawing the bifurcation diagram and the Lyapunov exponents for shorter intervals of the parameter r, for example for 0.6<r<0.7. Figure 6 clearly shows the appearance of chaos when $r \approx 0.64$.

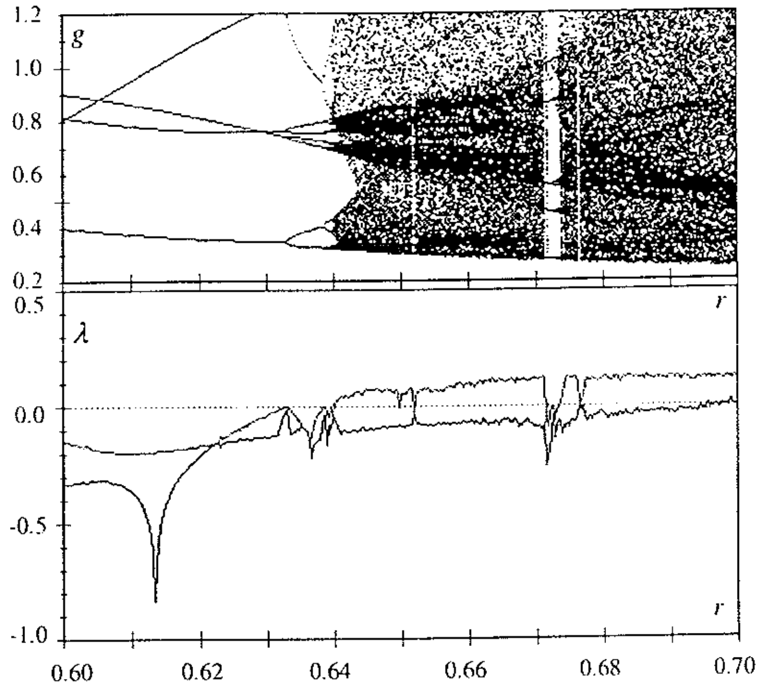

Figure 6: The behavior of goodwill with varying parameter $0.6<r<0.7$, when s=2, b=3 and q=1.2

In the chaos theory (see e.g. Lorenz, 1993; Medio, 1992; Seydel, 1994) two main routes to chaos, i.e. period doubling routes and quasiperiodic routes, are usually considered. The period doubling route to chaos is typical to one-dimensional systems and the quasiperiodic route to higher dimensional systems. The model (2.10) with time lag $\tau = 1$ is two-dimensional and the first bifurcation is a Hopf bifurcation. The Newhouse/Ruelle/Takens scenario (Lorenz, 1993) implies that after this standard Hopf bifurcation the behavior of the model is quasiperiodic or periodic untill one subsequent bifurcation to a two-frequency torus may happen. After these two bifurcations the system can come chaotic and chaos is more likely than a bifurcation to a 3-torus. The scenario refers, however to continuous-time systems, and in the discrete-time system (2.10) chaos seems to become through period doublings (Fig. 6). When the bifurcation diagrams and the Lyapunov exponents are drawn with other values of parameters s, b and q, the situation is the same: the route to chaos is through period doublings.

In the previous case the short time effect ($\tau = 1$) of advertising was studied. Now it is interesting to compare the short time effect to long time effects of advertising ($\tau = 3$ and $\tau = 5$, for example) to find out if the behavior of goodwill is qualitatively different.

Again, both the bifurcation diagrams and the values of the correlation dimension are drawn with the parameter r varying. With the help of the correlation dimension it is possible to determine whether the dynamical system is deterministic or stochastic. If the correlation dimension has a stationary value for increasing length m of the m-history vectors

$(x_i(t), x_i(t-1), \ldots, x_i(t-m+1))$ the system is deterministic. When this value permanently increases along with the embedding dimension m the system is stochastic.

In Figure 7 the time lag $\tau = 1$ so that the system and its bifurcation diagram is the same as in Figure. The values of the correlation dimension were plotted for five different values 5 up to the value m=18 of the embedding dimension and with varying values of the parameter r. The values of the correlation dimension for different embedding dimensions are overlapping to a great extent (especially for low values of r).

The dimension of the system is zero when the attractor is periodic, one in the case of a limit cycle or of a nearly a cycle and greater than one in the chaotic case. As can be seen from Figure 7, the dimensions of the attractors of the system are zero when the behavior of goodwill is stable, i.e. the attractors are fixed point attractors (and have the period one). When the limit point of the system becomes repelling the dimension changes to one and after that the values are between zero and one untill the dimensions of attractors approach the value two indicating chaos.

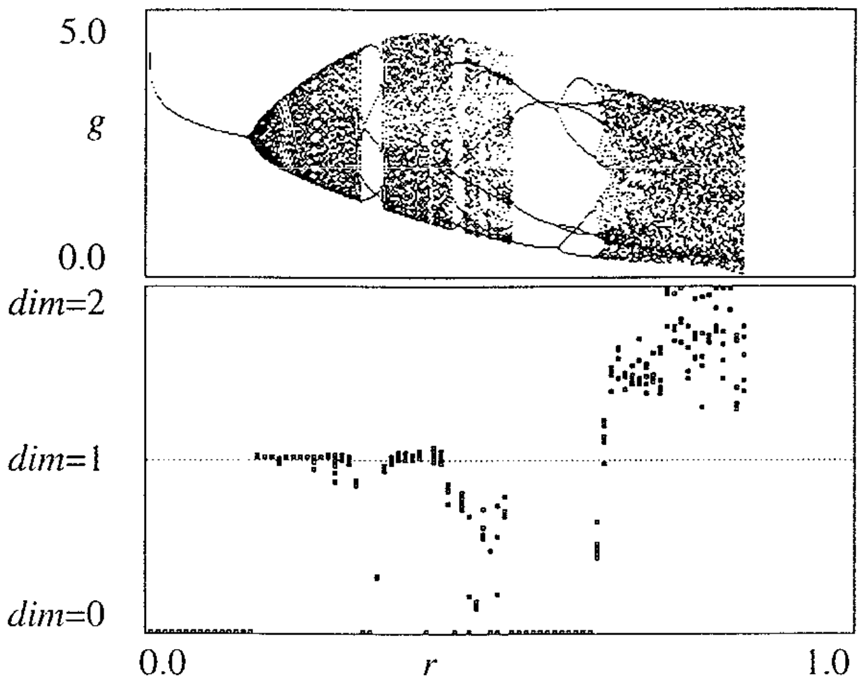

Figure 7: The behavior of goodwill with varying parameter 0<r<1,
when s=2, b=3, q=1.2 and $\tau = 1$

Figure 8 presents the case, where the time lag $\tau = 3$. The results are quite similar as in Figure 7. When the equilibrium goodwill, after having a stable phase in the beginning, becomes unstable, the dimensions of the attractors first increase from 0 to 1, then have values between 0 and 1, and lastly increase to a level indicating chaos. This means that in the neighbourhood of the Hopf bifurcation the dimensions of the attractors in the two cases above ($\tau = 1, \tau = 3$) are the same. This is typical for other time lags, too. Because the dimension on an attractor is smaller than the dimension of the state space it can be said that the model is qualitatively two-dimensional in spite of what is the value of the time delay $\tau > 1$. This is notable when determinig the stability conditions for fixed point attractors (for the first Hopf bifurcation) because the determination can be done mathematically in the same way as in the case of short time lag of advertising ($\tau = 1$).

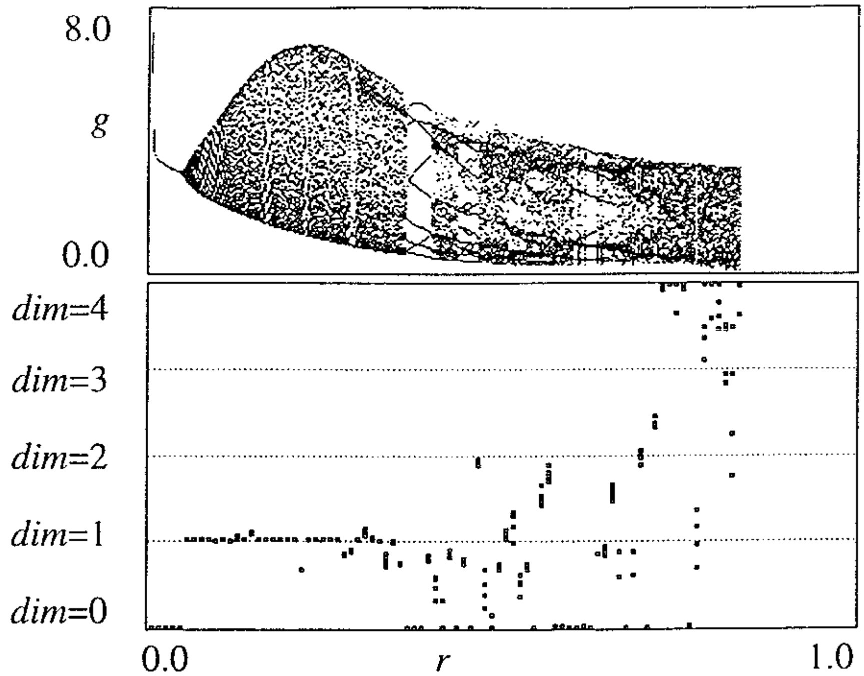

Figure 8: The behavior of goodwill with varying parameter $0<r<1$, when $s=2$, $b=3$, $q=1.2$ and $\tau=3$

4. Summary and Conclusions

Starting from the classical advertising capital model, the Nerlove-Arrow model, a time delayed feedback model between advertising and goodwill was introduced. It was assumed that the effect of advertising on the equilibrium goodwill is non-linear. With the short time effect (with time lag equal to one) the system becomes two-dimensional. Typical bifurcations in two-dimensional systems are Hopf bifurcations. The stability condition for a fixed point attractor, i.e. for the existence of the Hopf bifurcation was determined analytically. The correlation dimension technique was used to show numerically that the behavior of goodwill is qualitatively the same also when the dimension of the system is higher (the case of time lag greater than one). Also in this case the Hopf bifurcation occurs at the moment when the fixed point attractor becomes repelling.

Using bifurcation diagrams and Lyapunov exponents it was shown that after the Hopf bifurcation the cyclic behavior of goodwill is quite permanent, i.e. structually stable, with the increasing parameter value. When bifurcation diagrams and Lyapunov exponents are analyzed together the periodic, the nearly periodic and the chaotic areas of the behavior of goodwill can be seen. Chaos is clear for parameter values with a positive value for the largest Lyapunov exponent. Interesting is to notice that chaos becomes through period doubling bifurcations which is typical to one-dimensional systems. From correlation dimension it can be seen that before becoming chaotic the attractor is periodic, i.e. the system is one-dimensional.

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