2.5 *Variance decomposition and innovation accounting*

Consider the VAR(p) model

$$\Phi(L)\mathbf{y}_t = \epsilon_t,$$

where

$$\Phi(L) = \mathbf{I}_m - \Phi_1 L - \Phi_2 L^2 - \cdots - \Phi_p L^p$$

is the lag polynomial of order p with $m \times m$ coefficient matrices $\Phi_i,\ i = 1, \ldots p$.

Provided that the stationarity condition holds we may obtain a vector MA representation of $\mathbf{y}_t$ by left multiplication with $\Phi^{-1}(L)$ as

$$\mathbf{y}_t = \Phi^{-1}(L)\epsilon_t = \Psi(L)\epsilon_t$$

where

$$\Phi^{-1}(L) = \Psi(L) = \mathbf{I}_m + \Psi_1 L + \Psi_2 L^2 + \cdots .$$

The $m \times m$ coefficient matrices $\Psi_1, \Psi 2, \ldots$ may be obtained from the identity

$$\Phi(L)\Psi(L) = (\mathbf{I}_m - \sum_{i=1}^{p} \Phi_i L^i)(\mathbf{I}_m + \sum_{i=1}^{\infty} \Psi_i L^i) = \mathbf{I}_m$$

as

$$\Psi_j = \sum_{i=1}^{j} \Psi_{j-i} \Phi_i$$

with $\Psi_0 = \mathbf{I}_m$ and $\Phi_j = \mathbf{0}$ when $i > p$, by multiplying out and setting the resulting coefficient matrix for each power of L equal to zero. For example, start with $L_1^1 = L_1$:

$$-\Phi_1 L_1 + \Psi_1 L1 = (\Psi_1 - \Phi_1) L_1 \equiv 0$$

$$\Rightarrow \quad \Psi_1 = \Phi_1 = \Psi_0 \Phi_1 = \sum_{i=1}^{1} \Psi_{1-i} \Phi_i$$

Consider next L_1^2:

$$\Psi_2 L_1^2 - \Psi_1 \Phi_1 L_1^2 - \Phi_2 L_1^2 \equiv 0$$

$$\Rightarrow \quad \Psi_2 = \Psi_1 \Phi_1 + \Phi_2 = \sum_{i=1}^{2} \Psi_{2-i} \Phi_i$$

The result generalizes to any power L_1^j, which yields the transformation formula given above.

Now, since

$$\mathbf{y}_{t+s} = \Psi(L)\epsilon_{t+s} = \epsilon_{t+s} + \sum_{i=1}^{\infty} \Psi_i \epsilon_{t+s-i}$$

we have that the effect of a unit change in ϵ_t on $\mathbf{y}_{t+s}$ is

$$\frac{\partial \mathbf{y}_{t+s}}{\partial \epsilon_t} = \Psi_s.$$

Now the ϵ_t's represent shocks in the system. Therefore the Ψ_i matrices represent the model's response to a unit shock (or innovation) at time point t in *each* of the variables i periods ahead. Economists call such parameters dynamic multipliers.

The response of y_i to a unit shock in y_j is therefore given by the sequence below, known as the impulse response function,

$$\psi_{ij,1}, \psi_{ij,2}, \psi_{ij,3}, \ldots,$$

where $\psi_{ij,k}$ is the ijth element of the matrix Ψ_k $(i, j = 1, \ldots, m)$.

For example if we were told that the first element in ϵ_t changes by δ_1 at the same time that the second element changed by $\delta_2, \ldots,$ and the mth element by δ_m, then the combined effect of these changes on the value of the vector $\mathbf{y}_{t+s}$ would be given by

$$\Delta \mathbf{y}_{t+s} = \frac{\partial \mathbf{y}_{t+s}}{\partial \epsilon_{1t}} \delta_1 + \cdots + \frac{\partial \mathbf{y}_{t+s}}{\partial \epsilon_{mt}} \delta_m = \Psi_s \delta,$$

where $\delta' = (\delta_1, \ldots, \delta_m)$.

Generally an impulse response function traces the effect of a one-time shock to one of the innovations on current and future values of the endogenous variables.

Example: Exogeneity in MA representation

Suppose we have a bivariate VAR system such that x_t does not Granger cause y_t. Then we can write

$$\begin{pmatrix} y_t \\ x_t \end{pmatrix} = \begin{pmatrix} \phi_{11}^{(1)} & 0 \\ \phi_{21}^{(1)} & \phi_{22}^{(1)} \end{pmatrix} \begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} + \cdots$$
$$+ \begin{pmatrix} \phi_{11}^{(p)} & 0 \\ \phi_{21}^{(p)} & \phi_{22}^{(p)} \end{pmatrix} \begin{pmatrix} y_{t-p} \\ x_{t-p} \end{pmatrix} + \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}.$$

Then the coefficient matrices $\Psi_j = \sum_{i=1}^{j} \Psi_{j-i}\Phi_i$ in the corresponding MA representation are lower triangular as well (exercise):

$$\begin{pmatrix} y_t \\ x_t \end{pmatrix} = \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix} + \sum_{i=1}^{\infty} \begin{pmatrix} \psi_{11}^{(i)} & 0 \\ \psi_{21}^{(i)} & \psi_{22}^{(i)} \end{pmatrix} \begin{pmatrix} \epsilon_{1,t-i} \\ \epsilon_{2,t-i} \end{pmatrix}$$

Hence, we see that variable y does not react to a shock in x.

Ambiguity of impulse response functions

Consider a bivariate VAR model in vector MA representation, that is,

$$\mathbf{y}_t = \Psi(L)\epsilon_t \quad \text{with} \quad \mathsf{E}(\epsilon_t \epsilon_t') = \Sigma_\epsilon,$$

where $\Psi(L)$ gives the response of $\mathbf{y}_t = (y_{t1}, y_{t2})'$ to both elements of ϵ_t, that is, ϵ_{t1} and ϵ_{t2}.

Just as well we might be interested in evaluating responses of $\mathbf{y}_t$ to *linear combinations* of ϵ_{t1} and ϵ_{t2}, for example to unit movements in ϵ_{t1} and $\epsilon_{t2} + 0.5\epsilon_{t1}$. This may be done by defining new shocks $\nu_{t1} = \epsilon_{t1}$ and $\nu_{t2} = \epsilon_{t2} + 0.5\epsilon_{t1}$, or in matrix notation

$$\nu_t = Q\epsilon_t \quad \text{with} \quad Q = \begin{pmatrix} 1 & 0 \\ 0.5 & 1 \end{pmatrix}.$$

The vector MA representation of our VAR in terms of the new chocks becomes then

$$\mathbf{y}_t = \Psi(L)\epsilon_t = \Psi(L)Q^{-1}Q\epsilon_t =: \Psi^*(L)\nu_t$$

with

$$\Psi^*(L) = \Psi(L)Q^{-1}.$$

Note that both representations are observationally equivalent (they produce the same $\mathbf{y_t}$), but yield different impulse response functions. In particular,

$$\psi_0^* = \psi_0 \cdot Q^{-1} = \mathbf{I} \cdot Q^{-1} = Q^{-1},$$

which implies that single component shocks may now have contemporaneous effects on more than one component of $\mathbf{y_t}$. Also the covariance matrix of residuals will change, since

$$\mathsf{E}(\nu_t\nu_t') = \mathsf{E}(Q\epsilon_t\epsilon_t'Q') \neq \Sigma_\epsilon \quad \text{unless} \quad Q = \mathbf{I}.$$

But the fact that both representations are observationally equivalent implies that it is our own choice which linear combination of the ϵ_{ti}'s we find most useful to look at in the response analysis!

Orthogonalized impulse response functions

Usually the components of ϵ_t are contemporaneously correlated. For example in our VAR(2) model of the equity-bond data the contemporaneous residual correlations are

	FTA	DIV	R20	TBILL
FTA	1			
DIV	0.123	1		
R20	-0.247	-0.013	1	
TBILL	-0.133	0.081	0.456	1

If the correlations are high, it doesn't make much sense to ask "what if ϵ_{t1} has a unit impulse" with no change in ϵ_{t2} since both come usually at the same time. For impulse response analysis, it is therefore desirable to express the VAR in such a way that the shocks become orthogonal, (that is, the ϵ_{ti}'s are uncorrelated). Additionally it is convenient to rescale the shocks so that they have a unit variance.

So we want to pick a Q such that $\mathsf{E}(\nu_t\nu_t') = \mathbf{I}$. This may be accomplished by coosing Q such that $Q^{-1}Q^{-1\prime} = \Sigma_\epsilon$, since then

$$\mathsf{E}(\nu_t\nu_t') = \mathsf{E}(Q\epsilon_t\epsilon_t'Q') = \mathsf{E}(Q\Sigma_\epsilon Q') = \mathbf{I}.$$

Unfortunately there are many different Q's, whose inverse $\mathbf{S} = Q^{-1}$ act as "square roots" for Σ_ϵ, that is, $\mathbf{S}\mathbf{S}' = \Sigma_\epsilon$.

This may be seen as follows. Choose any orthogonal matrix $\mathbf{R}$ (that is, $\mathbf{R}\mathbf{R}' = \mathbf{I}$) and set $\mathbf{S}^* = \mathbf{S}\mathbf{R}$. We have then

$$\mathbf{S}^*\mathbf{S}^{*\prime} = \mathbf{S}\mathbf{R}\mathbf{R}'\mathbf{S}' = \mathbf{S}\mathbf{S}' = \Sigma_\epsilon.$$

Which of the many possible $\mathbf{S}$'s, respectively Q's, should we choose?

Before turning to a clever choice of Q (resp. $\mathbf{S}$), let us briefly restate our results obtained so far in terms of $\mathbf{S} = Q^{-1}$.

If we find a matrix $\mathbf{S}$ such that $\mathbf{S}\mathbf{S}' = \Sigma_\epsilon$, and transform our VAR residuals such that

$$\nu_t = \mathbf{S}^{-1}\epsilon_t,$$

then we obtain an observationally equivalent VAR where the shocks are orthogonal (i.e. uncorrelated with a unit variance), that is,

$$E(\nu_t\nu_t') = \mathbf{S}^{-1}E(\epsilon_t\epsilon_t')\mathbf{S}'^{-1} = \mathbf{S}^{-1}\Sigma_\epsilon\mathbf{S}'^{-1} = \mathbf{I}.$$

The new vector MA representation becomes

$$\mathbf{y}_t = \Psi^*(L)\nu_t = \sum_{i=0}^{\infty} \psi_i^*\nu_{t-i},$$

where $\psi_i^* = \psi_i\mathbf{S}$ ($m \times m$ matrices) so that $\psi_0^* = \mathbf{S} \neq \mathbf{I}_m$. The impulse response function of y_i to a unit shock in y_j is then given by the <u>othogonalised impulse response function</u>

$$\psi_{ij,0}^*, \psi_{ij,1}^*, \psi_{ij,2}^*, \cdots \; .$$

Choleski Decomposition and Ordering of Variables

Note that every orthogonalization of correlated shocks in the original VAR leads to contemporaneous effects of single component shocks ν_{ti} to more than one component of $\mathbf{y}_t$, since $\psi_0 = \mathbf{S}$ will not be diagonal unless Σ_ϵ was diagonal already.

One generally used method in choosing $\mathbf{S}$ is to use Cholesky decomposition which results in a lower triangular matrix with positive main diagonal elements for Ψ_0^*, e.g.

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} \psi_{11}^{*(0)} & 0 \\ \psi_{21}^{*(0)} & \psi_{22}^{*(0)} \end{pmatrix} \begin{pmatrix} \nu_{1,t} \\ \nu_{2,t} \end{pmatrix} + \Psi^*(1)\nu_{t-1} + \dots$$

Hence Cholesky decomposition of Σ_ϵ implies that the second shock $\nu_{2,t}$ does not affect the first variable $y_{1,t}$ contemporaneously, but both shocks can have a contemporaneous effect on $y_{2,t}$ (and all following variables, if we had choosen an example with more than two components). Hence the ordering of variables is important!

It is recommended to try various orderings to see whether the resulting interpretations are consistent. The principle is that the first variable should be selected such that it is the only one with potential immediate impact on all other variables. The second variable may have an immediate impact on the last $m-2$ components of $\mathbf{y}_t$, but not on y_{1t}, the first component, and so on. Of course this is usually a difficult task in practice.

Variance decomposition

The uncorrelatedness of the ν_t's allows to decompose the error variance of the s step-ahead forecast of y_{it} into components accounted for by these shocks, or innovations (this is why this technique is usually called innovation accounting). Because the innovations have unit variances (besides the uncorrelatedness), the components of this error variance accounted for by innovations to y_j is given by $\sum_{l=0}^{s-1} {\psi_{ij}^{*(l)}}^2$, as we shall see below.

Consider an orthogonalized VAR with m components in vector MA representation,

$$\mathbf{y}_t = \sum_{l=0}^{\infty} \psi^*(l)\nu_{t-l}.$$

The s step-ahead forecast for $\mathbf{y}_t$ is then

$$\mathsf{E}_t(\mathbf{y}_{t+s}) = \sum_{l=s}^{\infty} \psi^*(l)\nu_{t+s-l}.$$

Defining the s step-ahead forecast error as

$$\mathbf{e}_{t+s} = \mathbf{y}_{t+s} - \mathsf{E}_t(\mathbf{y}_{t+s})$$

we get

$$\mathbf{e}_{t+s} = \sum_{l=0}^{s-1} \psi^*(l)\nu_{t+s-l}.$$

It's i'th component is given by

$$e_{i,t+s} = \sum_{l=0}^{s-1}\sum_{j=1}^{m} \psi_{ij}^{*(l)}\nu_{j,t+s-l} = \sum_{j=1}^{m}\sum_{l=0}^{s-1} \psi_{ij}^{*(l)}\nu_{j,t+s-l}.$$

Now, because the shocks are both serially and contemporaneously uncorrelated, we get for the error variance

$$
\begin{aligned}
\mathbf{V}(e_{i,t+s}) &= \sum_{j=1}^{m}\sum_{l=0}^{s-1}\mathbf{V}(\psi_{ij}^{*(l)}\nu_{j,t+s-l}) \\
&= \sum_{j=1}^{m}\sum_{l=0}^{s-1}{\psi_{ij}^{*(l)}}^2\mathbf{V}(\nu_{j,t+s-l}).
\end{aligned}
$$

Now, recalling that all shock components have unit variance, this implies that

$$
\mathbf{V}(e_{i,t+s}) = \sum_{j=1}^{m}\left(\sum_{l=0}^{s-1}{\psi_{ij}^{*(l)}}^2\right),
$$

where $\sum_{l=0}^{s-1}{\psi_{ij}^{*(l)}}^2$ accounts for the error variance generatd by innovations to y_j, as claimed.

Comparing this to the sum of innovation responses we get a relative measure how important variable js innovations are in the explaining the variation in variable i at different step-ahead forecasts, i.e.,

$$
R_{ij,l}^2 = 100\frac{\sum_{l=0}^{s-1}{\psi_{ij}^{*(l)}}^2}{\sum_{k=1}^{m}\sum_{l=0}^{s-1}{\psi_{ik}^{*(l)}}^2}.
$$

Thus, while impulse response functions traces the effects of a shock to one endogenous variable on to the other variables in the VAR, variance decomposition separates the variation in an endogenous variable into the component shocks to the VAR.

Example. Let us choose in our example two orderings. One according to the feedback analysis

[(I: FTA, DIV, R20, TBILL)],

and an ordering based upon the relative timing of the trading hours of the markets

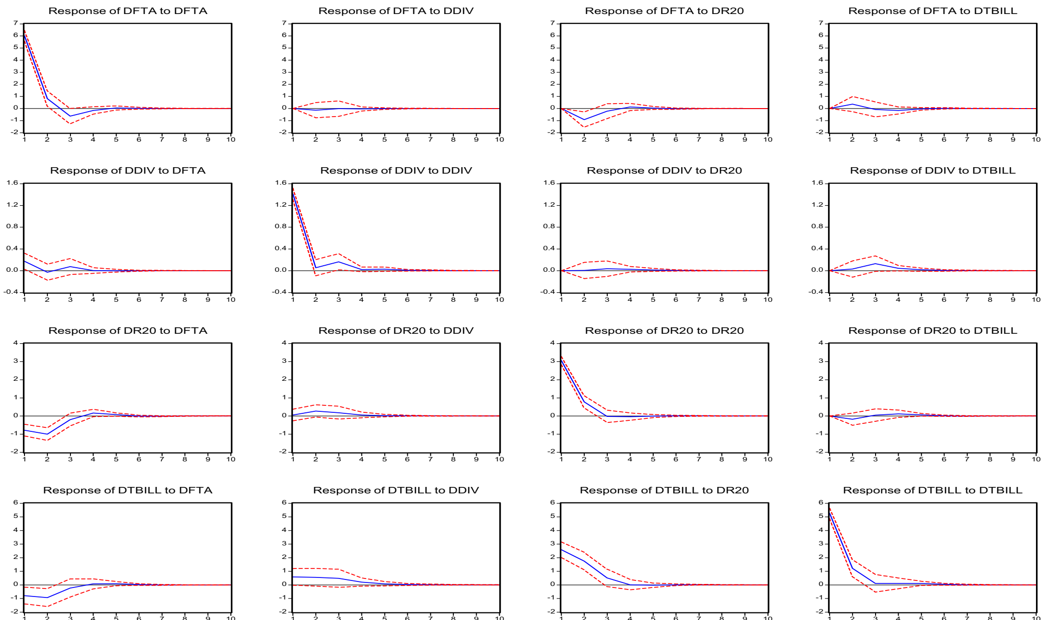
[(II: TBILL, R20, DIV, FTA)].

In EViews the order is simply defined in the Cholesky ordering option. Below are results in graphs with I: FTA, DIV, R20, TBILL; II: R20, TBILL DIV, FTA, and III: General impulse response function.

Impulse responses:

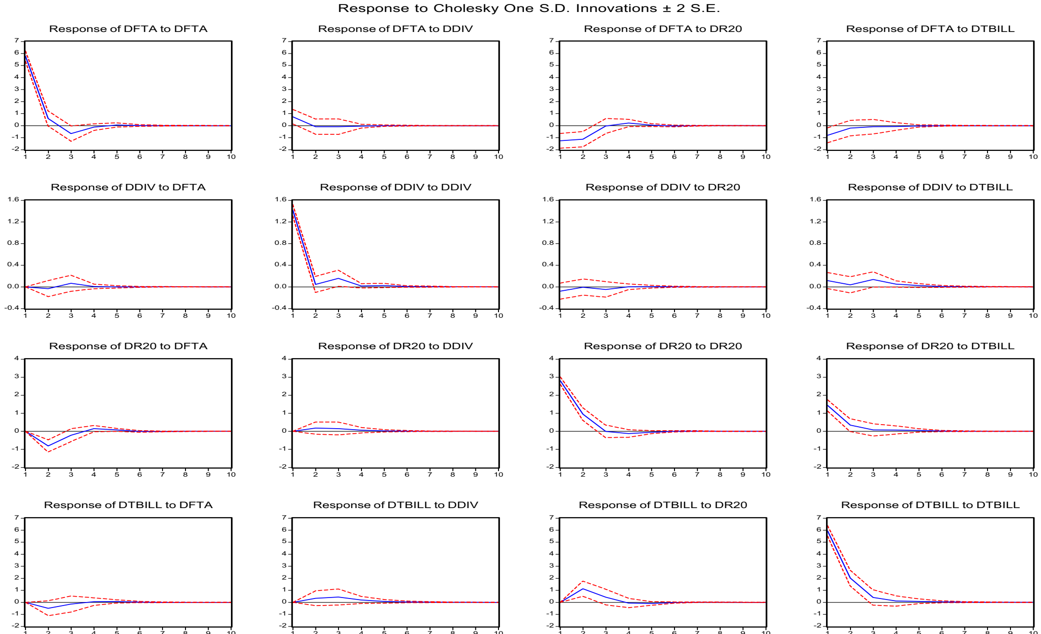
Order {FTA, DIV, R20, TBILL}

Response to Cholesky One S.D. Innovations ± 2 S.E.

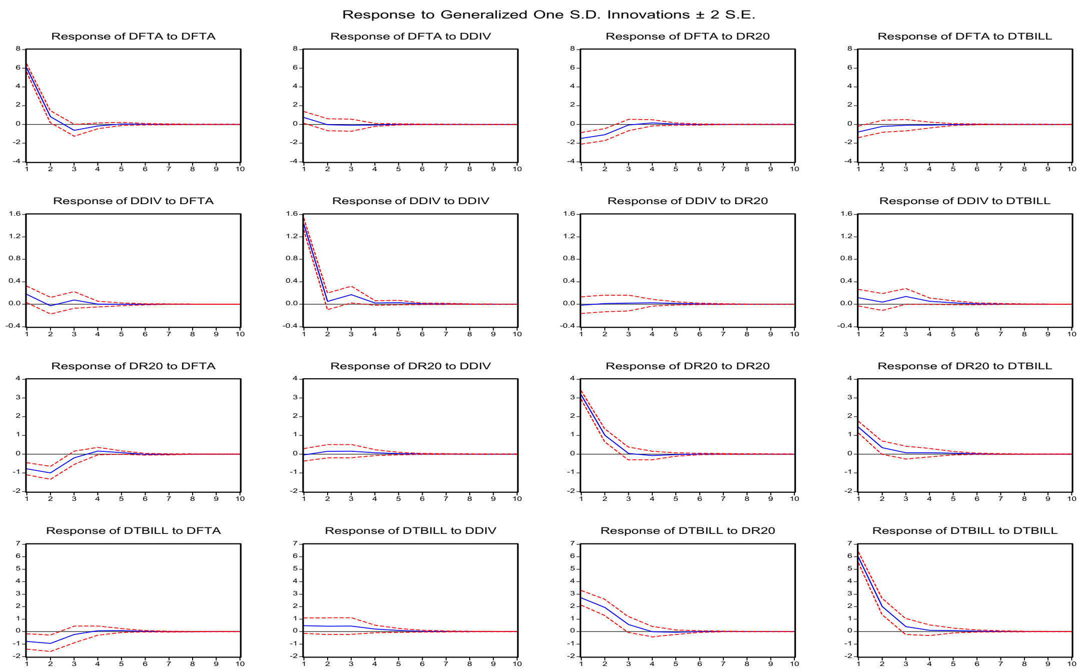


Order {TBILL, R20, DIV, FTA}

Response to Cholesky One S.D. Innovations ± 2 S.E.



Impulse responses continue:



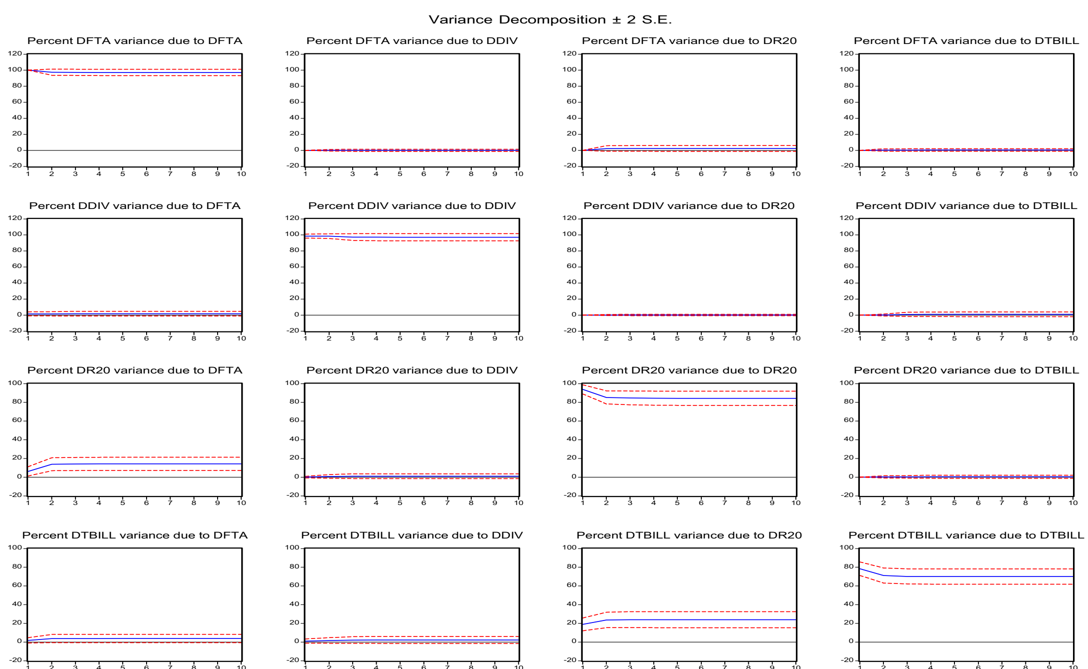
The general impulse response function are defined as[‡‡]

$$GI(j, \delta_i, \mathcal{F}_{t-1}) = \mathsf{E}[\mathbf{y}_{t+j}|\epsilon_{it} = \delta_i, \mathcal{F}_{t-1}] - \mathsf{E}[\mathbf{y}_{t+j}|\mathcal{F}_{t-1}].$$

That is difference of conditional expectation given an one time shock occurs in series j. These coincide with the orthogonalized impulse responses if the residual covariance matrix Σ is diagonal.

[‡‡] Pesaran, M. Hashem and Yongcheol Shin (1998). Impulse Response Analysis in Linear Multivariate Models, *Economics Letters*, 58, 17-29.

Variance decomposition graphs of the equity-bond data



On estimation of the impulse response coefficients

Consider the VAR(p) model

$$\Phi(L)\mathbf{y}_t = \epsilon_t,$$

with $\Phi(L) = \mathbf{I}_m - \Phi_1 L - \Phi_2 L^2 - \cdots - \Phi_p L^p$. Then under stationarity the vector MA representation is

$$\mathbf{y} = \epsilon + \Psi_1 \epsilon_{t-1} + \Psi_2 \epsilon_{t-2} + \cdots$$

When we have estimates of the AR-matrices Φ_i denoted by $\widehat{\Phi}_i$, $i = 1, \ldots, p$; the next problem is to construct estimates $\widehat{\Psi}_j$ for the MA matrices Ψ_j. Recall that

$$\Psi_j = \sum_{i=1}^{j} \Psi_{j-i} \Phi_i$$

with $\Psi_0 = \mathbf{I}_m$, and $\Phi_j = \mathbf{0}$ when $i > p$. The estimates $\widehat{\Psi}_j$ can be obtained by replacing the Φ_i's by their corresponding estimates $\widehat{\Phi}_i$.

Next we have to obtain the orthogonalized impulse response coefficients. This can be done easily, for letting $\mathbf{S}$ be the Cholesky decomposition of Σ_ϵ such that

$$\Sigma_\epsilon = \mathbf{S}\mathbf{S}',$$

we can write

$$\begin{aligned} \mathbf{y}_t &= \textstyle\sum_{i=0}^{\infty} \Psi_i \epsilon_{t-i} \\ &= \textstyle\sum_{i=0}^{\infty} \Psi_i \mathbf{S}\mathbf{S}^{-1} \epsilon_{t-i} \\ &= \textstyle\sum_{i=0}^{\infty} \Psi_i^* \nu_{t-i}, \end{aligned}$$

where

$$\Psi_i^* = \Psi_i \mathbf{S}$$

and $\nu_t = \mathbf{S}^{-1}\epsilon_t$. Then

$$\mathrm{Cov}(\nu_t) = \mathbf{S}^{-1}\Sigma_\epsilon \mathbf{S}'^{-1} = \mathbf{I}.$$

The estimates for Ψ_i^* are obtained by replacing Ψ_t with their estimates $\widehat{\Psi}_t$ and using Cholesky decomposition of $\hat{\Sigma}_\epsilon$.