Mathematics of Financial Derivatives: Exercise Sheet 3

1. Show that

$$\operatorname{Cov}\left(\int_0^t f(u)\,dW_u,\ \int_0^t g(u)\,dW_u\right) = \int_0^t \mathrm{E}\left(f(u)g(u)\right) du.$$

Hint: Show first that the covariance of the "elementary integrals" $I_n^{(1)} = \sum_{i=1}^n f(t_{i-1})\Delta W_{t_i}$ and $I_n^{(2)} = \sum_{i=1}^n g(t_{i-1})\Delta W_{t_i}$ reduces to a sum of the form $\sum_{i=1}^n \mathrm{E}\left(f(t_{i-1})g(t_{i-1})\right)\Delta t_i$. You may then use without proof that the mean square convergence of $I_n^{(1)}$ to $\int_0^t f(u)\,dW_u$ and $I_n^{(2)}$ to $\int_0^t g(u)\,dW_u$ implies

$$\mathrm{E}\left(\int_0^t f(u)\,dW_u \cdot \int_0^t g(u)\,dW_u\right) = \lim_{n\to\infty} \mathrm{E}(I_n^{(1)} \cdot I_n^{(2)}).$$

2. Show that

$$I_1(t) := \operatorname{lim} \sum_{i=1}^n W_{t_i}(W_{t_i} - W_{t_{i-1}}) = \frac{1}{2}(W_t^2 + t),$$

where "lim" denotes the limit in quadratic mean, W_t denotes standard Brownian motion, and $t_i = i \cdot (t/n)$.

Hint: $W_{t_i} = \frac{1}{2}(W_{t_i} + W_{t_{i-1}}) + \frac{1}{2}(W_{t_i} - W_{t_{i-1}})$