Mathematics of Financial Derivatives: Exercise Sheet 2

1. Let $Z(t) := \exp(\sigma W_t - \frac{\sigma^2}{2}t)$ with W_t denoting standard Brownian motion. Show that $Z(t)$ is a martingale.
   Hint: Show first that $Z(t+\Delta t)$ may be written as a product of the form:
   $$Z(t+\Delta t) = Z(t)\exp\left(\sigma \Delta W_t - \frac{\sigma^2}{2}\Delta t\right) \textit{ with } \Delta W_t := W_{t+\Delta t} - W_t.$$
   Use the independence of ΔW_t from $\mathcal{I}_t$ to replace the conditional expectation operator E_t with the unconditional expectation E. *In employing property i) you may use without proof that for any random variable $X \sim N(\mu, \sigma^2)$ we have $E(e^X) = e^{\mu + \frac{\sigma^2}{2}}$.*

2. Let $P(\Delta N_{t+h} = k) = \frac{(\lambda h)^k}{k!}e^{-\lambda h}, \quad k = 0, 1, \ldots$

   a) Show that $\mathrm{E}(\Delta N_{t+h}) = \lambda h$.
      Hint: Because ΔN_{t+h} may attain infinitely many values, you will have to calculate its expected value as an infinite sum:
      $$\mathrm{E}(\Delta N_{t+h}) = \sum_{k=0}^{\infty} k \cdot P(\Delta N_{t+h} = k)$$
      Introduce a new summation variable $i = k - 1$ and use the fact that $\sum_{i=0}^{\infty} \frac{(\lambda h)^i}{i!} = e^{\lambda h}$.

   b) Show that $\mathrm{V}(\Delta N_{t+h}) = \lambda h$.
      Hint: Calculate first $\mathrm{E}({\Delta N_{t+h}}^2) = \sum_{k=0}^{\infty} k^2 P(\Delta N_{t+h} = k)$ and introduce again a new summation variable $i = k - 1$. You will then notice that the sum may be split up into two components; the first representing $\mathrm{E}(\Delta N_{t+h}) = \lambda h$, and the second representing $\sum_{i=0}^{\infty} P(\Delta N_{t+h} = i) = 1$ because of (P2). *Use finally* (V1) *in order to obtain the variance.*

3. *Computer Exercise to be calculated in Excel:*
   You may hand in this exercise either on a floppy disk or by email.

   Suppose you wish to model the value of some financial asset as the sum of three independent components as follows. The first component is a jump process with an expected jump size of 1%=0.01 and one expected jump per month. The second component is also a jump process with an expected jump size of -5%=-0.05 and two expected jumps per year.

The third component is generalized Brownian motion with an annual drift of $\mu = 6\% = 0.06$ and an annual volatility of $\sigma = 15\% = 0.15$.

Generate sample paths of both the asset price and each of the underlying processes for one year at daily observation intervals, assuming 250 business days per year, and plot the result.

Hint: The fact that the three processes are independent implies that you need to generate one set of random variable for each component seperately. You obtain a jump process with an expected jump size of x from a Poisson process with an expected jump size of 1 simply by multiplying either its increments or the process itself by x. Use the scaling property of Brownian Motion to obtain daily increments in the third component.