Mathematics of Financial Derivatives: Exercise Sheet 1

1. Let $P : \Omega \to I\!R$ be a probability measure and $A, B \subset \Omega$.

   a) Show that $P(A^C) = 1 - P(A)$.
   Hint: Note that the sample space may be written as the union of any set with its complement, that is $\Omega = A \cup A^C$, and use the fact that the intersection of any set with its complement is the empty set, that is $A \cap A^C = \emptyset$, such that you may apply (P3).

   b) Show that $P(\emptyset) = 0$.
   Hint: Note that $\Omega^C = \emptyset$ and apply a).

   c) Show that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
   Hint: Note that $A \cup B$ and B may be written as unions of non-intersecting sets as shown below and apply (P3):

   $$A \cup B = A \cup (A^C \cap B), \qquad B = (A \cap B) \cup (A^C \cap B).$$

2. Let $\mathcal{F} = \{F_1, \ldots, F_n\}$ be any algebra.

   a) Show that the empty set $\emptyset$ is always contained in this algebra.
   Hint: Note that $\emptyset = \Omega^C$ and use (A2).

   b) Show that for any F_i and F_j contained in $\mathcal{F}$ (that is, $F_i, F_j \in \mathcal{F}$), the intersection of F_i and F_j will be contained in $\mathcal{F}$ as well (that is, $F_i \cap F_j \in \mathcal{F}$).
   Hint: Note that $(F_i \cap F_j)^C = F_i^C \cup F_j^C$ and apply (A2),(A3).

3. A stock price process $S = \{S(t) : t = 0, 1, 2\}$ starts with $S_0 = 30\$$. After that, the stock price is equally likely to appreciate by 30\$, depreciate by 30\$, or stay constant. A stock price of 0\$ is equally likely to appreciate by 30\$ or stay constant.

   a) Draw the corresponding information tree containing all possible stock prices S_t and transition probabililities $P(S_t|S_{t-1})$ for $t = 0, 1, 2$.

   b) Find the algebras $\mathcal{F}_t$ and their generating partitions $\mathcal{P}_t$ reflecting the maximum information available at $t = 0, 1, 2$.
   Hint: It's easiest to first find the generating partitions $\mathcal{P}_t$ and to construct the corresponding algebras $\mathcal{F}_t$ afterwards. $\mathcal{F}_2$ is pretty long. You don't have to write it down explicitely, just indicate how you would construct it.

c) Use the law of total probability in order to find the probability densities and unconditional expectations of both S_1 and S_2.

d) Calculate the conditional expectations $E(S_2|\mathcal{F}_t)$ for $t = 0, 1, 2$.

e) Verify that $E(S_2|\mathcal{F}_0) = E(E(S_2|\mathcal{F}_1)|\mathcal{F}_0)$.

f) Is S any of the following: a martingale, a supermartingale, or a submartingale? Why so?