ANALYSIS OF FINANCIAL TIME SERIES: EXERCISE SHEET 2

1. Considering T observations of a time series $y_1, \ldots, y_T$ as a random sample of a random variable Y, yields for the likelihood function the joint distribution of the sample $L(\theta) = f(y_1, \ldots, y_T|\theta)$, where the yet unknown parameter vector θ is to be determined by maximazation of L. This approach is known as estimation by maximum likelihood.

   Using the chain rule for conditional probabilities (which applies to probability densities as well), the joint density may be written as

   $$f(y_1, \ldots, y_T|\theta) = f(y_T|\mathcal{F}_{T-1};\theta)\times f(y_{T-1}|\mathcal{F}_{T-2};\theta)\times\cdots\times f(y_2|\mathcal{F}_1;\theta)\times f(y_1;\theta),$$

   where the information set $\mathcal{F}_t$ collects all available information up to time t, e.g. $\mathcal{F}_t = \{y_1, \ldots, y_t\}$ for a univariate time series. The log-likelihood function is obtained as the natural logarithm of the likelihood function:

   $$\ell(\theta) = \log L(\theta) = \sum_{t=1}^{T} \log f(y_t|\mathcal{F}_{t-1};\theta)$$

   Applying this approach to the observations $y_t = \mathbf{x}_t'\beta + u_t$ with ARCH residuals $u_t|\mathcal{F}_{t-1} \sim N(0, h_t)$ such that $y_t|\mathcal{F}_{t-1} \sim N(\mathbf{x}_t'\beta, h_t)$, yields the log-likelihood function given in the lecture notes. More generally, the residuals of an ARCH process may be defined to follow some other symmetric distributions, which are not necessarily normal. As an example, consider the scaled Student t-distribution with ν degrees of freedom, mean μ, variance $\sigma^2 = \frac{\nu}{H(\nu-2)}$, and probability density function

   $$g(y) = \frac{\Gamma\left((\nu+1)/2\right)}{\sqrt{\pi}\Gamma(\nu/2)}\nu^{\nu/2}\left[\nu + H(y-\mu)^2\right]^{-\frac{\nu+1}{2}}\sqrt{H},$$

   where $\Gamma(\cdot)$ denotes the gamma function.

   a) Show that the log-likelihood function of an ARCH specification $y_t = \mathbf{x}_t'\beta + u_t$ with $u_t|\mathcal{F}_{t-1} \sim$ scaled Student t$(\mu = 0, \sigma_t^2 = \frac{\nu}{H_t(\nu-2)})$ may be written as:

   $$\ell(\theta) = \sum_{t=1}^{T}\left\{-\frac{1}{2}\log\left(\frac{\pi(\nu-2)\Gamma(\nu/2)^2}{\Gamma\left((\nu+1)/2\right)^2}\right) - \frac{1}{2}\log\sigma_t^2 - \frac{\nu+1}{2}\log\left[1 + \frac{(y_t - \mathbf{x}_t'\beta)^2}{\sigma_t^2(\nu-2)}\right]\right\}$$

   Hint: $y_t|\mathcal{F}_{t-1} \sim$ *scaled Student t*$(\mu = \mathbf{x}_t'\beta, \sigma_t^2 = \frac{\nu}{H_t(\nu-2)})$.

b) Show that the likelihood function of an ARCH model with conditionally Student t distributed residuals as above converges to the likelihood function of an ARCH model with conditionally normally distributed residuals for $\nu \to \infty$.
Hint:

$$\frac{\Gamma\left(z+\frac{1}{4}\right)}{\Gamma\left(z-\frac{1}{4}\right)} \stackrel{z\to\infty}{\longrightarrow} \sqrt{z}, \qquad \left(1+\frac{x}{r}\right)^{-r} \stackrel{r\to\infty}{\longrightarrow} e^{-x}$$

2. Consider the GARCH(1,1) model $h_t = \omega + \alpha u_{t-1}^2 + \delta h_{t-1}$ with $\omega > 0$, $\alpha > 0$, $\delta \geq 0$, and $\alpha + \delta < 1$.

a) Show that

$$\sigma_u^2 := \operatorname{var}(u_t) = \frac{\omega}{1-\alpha-\delta}.$$

Hint: Show first that $n+1$ applications of the law of iterated expectations yield an expression of the form

$$\operatorname{var}(u_t) = \omega\left(1 + (\alpha+\delta) + \ldots + (\alpha+\delta)^n\right) + (\alpha+\delta)^{n+1} E({u_{t-n-1}}^2)$$

and take the limit $n \to \infty$.
Alternatively you may wish to write down the ARMA representation of the model and use the stationarity of u_t^2.

b) Show that

$$\mathrm{E}(u_t^4) = 3\sigma_u^4 \cdot \frac{1-(\alpha+\delta)^2}{1-(\alpha+\delta)^2-2\alpha^2}$$

provided that $(\alpha+\delta)^2 + 2\alpha^2 < 1$.
Hint: Show first that $n+1$ applications of the law of iterated expectations yield an expression of the form

$$\begin{aligned}\mathrm{E}(u_t^4) =& 3\omega^2 \frac{1+(\alpha+\beta)}{1-(\alpha+\beta)}\left(1 + (3\alpha^2+2\alpha\delta+\delta^2) + \ldots + (3\alpha^2+2\alpha\delta+\delta^2)^n\right) \\ &+ (3\alpha^2+2\alpha\delta+\delta^2)^{n+1}\mathrm{E}({u_{t-n-1}}^4)\end{aligned}$$

and take the limit $n \to \infty$.

c) Show that the k-step ahead forecast of the variance is

$$\mathrm{E}_t(u_{t+k}^2) = \sigma_u^2 + (\alpha+\delta)^{k-1}(h_{t+1} - \sigma_u^2),$$

which implies that long horizon forecasts converge to the unconditional variance $\sigma_u^2 = \omega/(1-\alpha-\delta)$.
Hint: Use the law of iterated expectations.