ANALYSIS OF FINANCIAL TIME SERIES: EXERCISE SHEET 1

1. Reconsider the coin tossing example in the note on conditional distributions, where this time the coin is biased such that the probability of head up is 3/4.

   a) Derive both the unconditional probability distribution of $X = X_1 + X_2$ and the conditional distribution of $X|X_1$.

   b) Calculate the conditional expectation $E(X|\mathcal{F}_1) = E(X|X_1)$ and verify that the law of iterated expectations holds.

2. Consider the general ARCH(q) process $h_t = \omega + \sum_{i=1}^{q} \alpha_i u_{t-1}^2$ with $\omega > 0$, $\alpha_i \geq 0$ for all i, $\sum_{i=1}^{q} \alpha_i < 1$, and $u_t|\mathcal{F}_{t-1} \sim N(0, h_t)$. The idea of this exercise is to show that u_t is again white noise, just like for ARCH(1).

   a) Show that $\mathrm{E}(u_t) = 0$.
   Hint: Use the law of iterated expectations.

   b) Show that $\mathrm{var}(u_t) = \omega/(1 - \sum_{i=1}^{q} \alpha_i)$.
   Hint: Show first that repeated application of the law of iterated expectations yields an expression of the form

$$\mathrm{var}(u_t) = \omega\left(1 + \sum_{i=1}^{q} \alpha_i + \ldots + \left(\sum_{i=1}^{q} \alpha_i\right)^n\right) + \left(\sum_{i=1}^{q} \alpha_i\right)^n \left(\sum_{i=1}^{q} \alpha_i E(h_{t-i})\right)$$

   and take the limit $n \to \infty$.
   Alternatively, and simpler, you may wish to write down the AR(q) representation of u_t^2, apply the expectation operator, and use the stationarity of u_t^2.

   c) Show that $\mathrm{cov}(u_t, u_{t+k}) = 0$ for all $k \neq 0$.
   Hint: Use the law of iterated expectations.

3. The idea of this exercise is to compare the moments of a NID$(0, \sigma^2)$ process with the residuals of an ARCH(1) process. Consider first the m'th moment $M_m = \int_{-\infty}^{\infty} x^m f(x)\, dx$ of a NID$(0, \sigma^2)$ process with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/(2\sigma^2)}.$$

   For our purpose it is practical to substitute $y = x/\sigma$, such that

$$M_m = \frac{\sigma^m}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^m e^{-y^2/2}\, dy.$$

a) Show that all uneven moments are zero, that is, $M_m = 0$ for $m = 2n+1$, $n = 0, 1, 2, \ldots$, implying both zero mean and zero skewness.
Hint: Split up the integral into two integrals with limits $0, \pm\infty$ *and apply the symmetry property of the integrand.*

b) Show that the variance M_2 of the process is σ^2.
Hint: Use partial integration ($\int uv' = uv - \int u'v$) *with* $u = y$ *and* $v' = -ye^{-y^2/2}$.

c) Show that the kurtosis M_4 of the process is $3\sigma^4$.
Hint: Use partial integration and apply your result obtained in b).

Let now $u_t \sim$ARCH(1), with $u_t|u_{t-1} \sim N(0, h_t)$, where $h_t = \omega + \alpha u_{t-1}^2$, $\omega > 0$ and $0 \leq \alpha < \sqrt{1/3}$.

d) Show that all uneven moments of u_t are zero: $\mathrm{E}(u_t^{2n+1}) = 0$, implying a symmetrical distribution with both zero mean and skewness.
Hint: There is no integration required. Just use the law of iterated expectations, the fact that u_t *is conditionally normally distributed, and your result from* a).

e) Show that

$$\mathrm{E}[u_t^4] = 3\frac{\omega^2}{(1-\alpha)^2} \cdot \frac{1-\alpha^2}{1-3\alpha^2},$$

which implies that u_t has a larger kurtosis than a normal distribution with the same variance.
Hint: Show first that $n+1$ *applications of the law of iterated expectations yield an expression of the form*

$$E[u_t^4] = 3\omega^2 \frac{1+\alpha}{1-\alpha}\left(1 + 3\alpha^2 + \ldots + \left(3\alpha^2\right)^n\right) + \left(3\alpha^2\right)^{n+1} E({u_{t-n-1}}^4)$$

and take the limit $n \to \infty$.