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OPTIMAL MAINTENANCE POLICY AND SALE DATE FOR A MACHINE WITH RANDOM DETERIORATION AND SUBJECT TO RANDOM CATASTROPHIC FAILURE

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OPTIMAL MAINTENANCE POLICY AND SALE DATE FOR A MACHINE WITH RANDOM DETERIORATION AND SUBJECT TO RANDOM CATASTROPHIC FAILURE

Ilkka Virtanen †

Abstract - The problem of both providing optimal maintenance for a machine during its service life and simultaneously selecting an optimal sale date for the machine is considered from a control-theoretic viewpoint. Both the deterioration and the life time of the machine are considered as random processes. The salvage value and the deterioration rate of the machine are treated as state variables and the maintenance expenditure as a control variable. The stochastic maximum principle is applied to derive the conditions for the optimal maintenance policy and for the optimal planned sale date which maximize the expected net present value of the machine, the performance index of the problem. An explicit solution is found analytically for the problem in the special case when some of the random processes of the model are independent of time and thus simply random variables. The case of one particular life-time probability distribution, namely the exponential case, is analyzed in full detail. The parameter of the distribution, i.e. the failure rate of the machine, is shown to have an interesting and important economic interpretation. The failure rate represents a risk premium which can be used to adjust both the mean production rate and the discount rate to the level of a certainty - equivalent problem.

1 Introduction

When a machine is used for production purposes and it ages, it suffers one of the two fates - either there is a gradual deterioration or a sudden failure. The first situation means more frequent repairs, a decrease in performance of the machine etc., the machine produces decreasing net receipts over time. This deterioration can be partially offset via preventive maintenance, and there also exists, of course, the possibility of selling the machine at any

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time, although its salvage value declines over time. The second situation makes the machine unusable for production and it has to be junked and replaced by a new machine. The problem of how the maintenance policy and sale date for the machine should be simultaneously optimized (the "Boiteaux problem") is the subject of the present paper.

Since Näslund [7] had initiated the control theory approach to simultaneous maintenance and sale date optimization, Thompson [10] first formulated for the problem an explicit model and solved it in detail. Thompson's model is completely deterministic, the machine cannot fail and its deterioration with age obeys a given mathematical law. Other formulations for the deterministic problem have been later presented e.g. by Arora and Lele [3], Bensoussan et al. [5] and Scott and Jefferson [9]. Kamien and Schwartz [6] developed a stochastic model where the failure part of the problem was included but the degradation of the machine with age was not considered. Due to Alam and Sarma [2] is a model where both the deterioration and the failing of the machine have been incorporated in a single model. The deterioration is taken as deterministic, whereas the machine is subject to random failure. However, in model [2] only the maintenance policy is optimized, the selling of a still operable machine is not considered. This may lead to an unprofitable use of the machine and to an improper optimum for the problem. The author of this paper has recently presented a generalized model [11] where both the maintenance policy and the sale date of the machine are optimized and thus a better result for the objective is obtained.

In this paper we consider a model where the random nature of both the deterioration of the machine over time and the life time of the machine are taken into account. Alam et al. [1] have earlier presented a model where both these aspects have been included, but this model has the same disadvantage as model [2]: in lack of the option of selling a still operable, but almost worthless and highly unprofitable machine, the model may lead to an improper optimum, even to such a use of the machine that does not pay for itself. We generalize model [1] and make it more realistic by taking also the sale

date of the machine (called the planned sale date due to the possibility of machine failing before that time) as a tool of optimization. We organize the contents of the paper as follows.

In section 2 we briefly describe Thompson's model [10] as well as the model of Alam et al. [1] to which the problem studied here is most closely related. In section 3 we extend and generalize the models [10] and [1] by formulating our own model. In section 4 we apply the stochastic maximum principle in order to derive the necessary conditions for the optimal maintenance policy and for the optimal sale date. In section 5 we consider in more detail the special case when an analytic solution for the problem is possible, and derive the solution. In section 6 we present the solution of the problem for a particular probability distribution of time to failure, viz., for the exponential distribution, and comment on the economic interpretations of the randomness in this connection. Finally, in section 7 we illustrate the results obtained with the help of a simple numerical example.

2 The models of Thompson and Alam et al.

Thompson considers the following deterministic problem: find the optimal maintenance policy $u(t)$ and the optimal sale date T for a machine to maximize the present value $V(T)$ of the machine given by

$$(1) \qquad V(T) = S(T)\exp(-rT) + \int_0^T [pS(t) - u(t)]\exp(-rt)dt$$

where the salvage value $S(t)$ is affected by the deterioration factor and the amount and the effectiveness of preventive maintenance according to the differential equation

$$(2) \qquad \frac{dS(t)}{dt} = -\delta(t) + f(t)u(t), \quad S(0) = S_0.$$

In (1) and (2) r is the rate of interest (the discount rate), $\delta(t)$ is the deterioration rate, $f(t)$ is the maintenance effectiveness function, and p is the (constant) production rate. The maintenance function u(t) (= money spent at time t over and above the minimum spent on necessary repairs) is the control variable satisfying for all t ($0 \le t \le T$) the requirement

(3) $$0 \le u(t) \le U,$$

and V(t) and S(t) are the state variables. An application of the maximum principle gives the following optimal maintenance policy $u^*(t)$

(4) $$u^*(t) = \begin{cases} U, \text{ if } f(t) > g_T(t) \\ \text{arbitrary } \in [0,U], \text{ if } f(t) = g_T(t) \\ 0, \text{ if } f(t) < g_T(t), \end{cases}$$

where we have denoted

(5) $$g_T(t) = r/\{p - (p-r)\exp[-r(T-t)]\}.$$

With Thompson's assumptions: δ, f and u are piecewise continuous, δ is non-decreasing, and f is non-increasing, the optimal maintenance policy becomes one of the following three types:

1^0 $u^*(t) = U$ for all $t \in [0,T]$

2^0 $u^*(t) = 0$ for all $t \in [0,T]$

3^0 $$u^*(t) = \begin{cases} U \text{ for } t \in [0,T') \\ \text{arbitrary } \in [0,U] \text{ for } t = T' \\ 0 \text{ for } t \in (T',T] \end{cases}$$

In the policy 3^0, T' is the solution of the equation $f(T') = g_T(T')$. The optimal sale date T is obtained as the solution of the equation

(6) $$S(T) = \{\delta(T) - [f(T) - 1]u^*(T)\}/(p-r).$$

The simultaneous determination of $u^*(t)$ and T may be carried out by a trial and error procedure utilizing (2) and (4) to (6).

Alam et al. [1] take Thompson's model as the starting point and begin by modelling the deterioration process as a random process whereas machine failing is not considered in the first phase. We call this model Alam I. The deterioration rate is considered as a stochastic process, which is assumed to be governed by the stochastic differential equation (our notation is slightly different from the original one)

(7) $$\frac{d\underline{\delta}(t)}{dt} = \underline{\alpha}(t) - \beta(t)u(t)$$

with the stochastic boundary condition

(8) $$\underline{\delta}(0) = \underline{\delta}_0.$$

The stochastic processes $\underline{\alpha}(t)$ and $\underline{\delta}(t)$ as well as the random variable $\underline{\delta}_0$ are assumed to be defined on a certain sample space Ω, the probability measure joining with Ω being P (generally speaking, we use the notation $\underline{z}$ or $\underline{z}(t)$ to indicate that the quantity z or z(t) is a random variable or a stochastic process, respectively). Because of (1) and (2), also the salvage value S(t) and the present value V(T) to be maximized will now be stochastic processes on Ω, denoted by $\underline{S}(t)$ and $\underline{V}(T)$, respectively.

The problem (Alam I) is now to choose $u^*(t)$ and T so as to maximize

(9) $$\bar{V}(T) = E\{\underline{V}(T)\} = \int_\Omega \underline{V}(T)dP,$$

where

(10) $$\underline{V}(T) = \underline{S}(T)\exp(-rT) + \int_0^T [p(t)\underline{S}(t) - u(t)]\exp(-rt)dt$$

subject to the state equations

(11) $$\frac{d\underline{S}(t)}{dt} = -\underline{\delta}(t) + f(t)u(t),\ 0 \le t \le T;\ \underline{S}(0) = S_0$$

and

(12) $$\frac{d\underline{\delta}(t)}{dt} = \underline{\alpha}(t) - \beta(t)u(t),\ 0 \le t \le T;\ \underline{\delta}(0) = \underline{\delta}_o$$

and to the control constraint

(13) $$0 \le u(t) \le U,\ 0 \le t \le T.$$

Applying the stochastic maximum principle, the solution of the problem can be derived. An analytic solution is possible in the special case when $\underline{\alpha}(t)$ and $\underline{p}(t)$ don't depend on time, they are simply random variables: $\underline{\alpha}(t) \equiv \underline{\alpha}$ and $\underline{p}(t) \equiv \underline{p}$. The optimal maintenance policy becomes

(14) $$u^*(t) = \begin{cases} U, \text{ if } f(t) > \dfrac{r - \beta(t)\{\frac{\bar{p}}{r}[1-\exp\{-r(T-t)\}] - (\bar{p}-r)(T-t)\exp\{-r(T-t)\}\}}{\bar{p} - (\bar{p}-r)\exp\{-r(T-t)\}} \\ 0, \text{ otherwise} \end{cases}$$

where ($\bar{\ }$) denotes the mean value of the random variable (here the mean production rate). The optimal sale date T is found from the equation

(15) $$\hat{S}(T) = \{\hat{\delta}(T) - [f(T) - 1]u^*(T)\}/(\bar{p}-r)$$

where ($\hat{\ }$) denotes the predicted estimate, e.g. the mean value, of the random variable.

In the second phase Alam et al. [1] take also the probability of machine failure into account and derive now the optimal maintenance policy for a machine with random deterioration and which is subject to random catastrophic failure (whereas the sale date of the machine is not considered, the machine is kept as long as it is operable). We call this model Alam II. Let τ denote the random life time of the machine and let $p_\tau(t;u(s),\ 0 \le s \le t)$, $P_\tau(t;u(s),\ 0 \le s \le t)$ and $Q_\tau(t;u(s),\ 0 \le s \le t)$ denote its density function, cumulative distribution function and complementary distribution (or reliability) function, respectively. Further, let $p_\tau(t;u)$, $P_\tau(t;u)$ and $Q_\tau(t;u)$ compactly represent these quantities. Assuming the deterioration process and failure process mutually independent, the following model (Alam II) can be stated: choose an optimal policy $u^*(t)$ so as to satisfy the constraints (11) to (13) and to maximize the expectation

(16) $$E_p\{E_\tau[\underline{V}(\tau)]\} = E_p\{\underline{V}_F\} = \int_\Omega \underline{V}_F dP$$

where

(17) $$\underline{V}_F = E_\tau[\underline{V}(\tau)] = \int_0^\infty \{[Q_\tau(t;u)\underline{p}(t) + p_\tau(t;u)]\underline{S}(t) - Q_\tau(t;u)u(t)\}\exp(-rt)dt$$

is the expectation of $\underline{V}(\tau)$, $\underline{V}(\tau)$ being the quantity (10) with T considered as the random variable τ and the expectation being with respect to τ. The second expectation E_p in (16) is with respect to the probability measure P.

Again, the solution of the problem can be found via application of the stochastic maximum principle. An analytic solution becomes possible when $\underline{\alpha}(t)$ and $\underline{p}(t)$ are simply random variables: $\underline{\alpha}(t) \equiv \underline{\alpha}$ and $\underline{p}(t) = \underline{p}$, and when failure probability is independent of maintenance: $p_\tau(t;u) = p_\tau(t)$ and, hence, $Q_\tau(t;u) = Q_\tau(t)$. In the case of an exponentially distributed life time, for example, the following optimal policy is obtained:

(18) $$u^*(t) = \begin{cases} U, \text{ if } f(t) > \dfrac{r+\sigma}{\bar{p}+\sigma} - \dfrac{\beta(t)}{r+\sigma} \\ 0, \text{ otherwise.} \end{cases}$$

In (18), σ is the parameter of the distribution giving the constant failure rate of the machine.

3 The generalized model

Both the models Alam I and Alam II presented in the previous section contain a certain unsatisfactory aspect in their formulation. The former represents an unrealistic situation in practice by assuming the machine as unbreakable, the latter may lead to an unprofitable use of the machine by forcing the owner to use the machine until it fails, regardless of its ever declining quality and productivity. In this paper, we provide for the problem a generalized formulation in which the above disadvantages are not included. We seek a planned sale date T and a planned maintenance policy $u^*(t)$, $0 \leq t \leq T$, for the machine until it is sold or it fails and must be junked, whichever comes first, so as to maximize the expected present value of the machine. The machine is assumed to suffer random deterioration as well as to be subject to random catastrophic failure.

The state equations considered are now

$$(19) \qquad \frac{d\underline{S}(t)}{dt} = -\underline{\delta}(t) + f(t)u(t), \ \underline{S}(0) = S_0$$

and

$$(20) \qquad \frac{d\underline{\delta}(t)}{dt} = \underline{\alpha}(t) - \beta(t)u(t), \ \underline{\delta}(0) = \underline{\delta}_0,$$

where we have used the same notation as in connection with the model Alam I, cf. equations (11) and (12). Let T denote the planned sale date of the machine, i.e. T is the time at which the machine will be sold provided it has not failed and been junked before that time. Also the maintenance function u(t) now represents the planned maintenance policy which will be obeyed as long as the machine is working (for a failed machine we have, of course, u(t) = 0). After all, we can set the usual control constraint

$$(21) \qquad 0 \leq u(t) \leq U, \ 0 \leq t \leq T.$$

As before, let τ denote the random life time of the machine and let $p_\tau(t;u)$, $P_\tau(t;u)$ and $Q_\tau(t;u)$ again compactly denote the density function, the cumulative distribution function and the reliability function of the random variable τ, respectively.

The present value of the machine at time t is, provided the machine is still operable, according to (10),

$$(22) \qquad \underline{V}(t) = \underline{S}(t)\exp(-rt) + \int_0^t [\underline{p}(t)\underline{S}(t) - u(t)]\exp(-rt)dt.$$

Let $\underline{V}_0(T)$ denote the present value which will be really obtained when the planned sale date of the machine is T. By assuming the junk value of the machine equal to its salvage value at the failure time, we get

$$(23) \qquad \underline{V}_0(T) = \begin{cases} \underline{V}(T), & \text{if } \tau \geq T \\ \underline{V}(\tau), & \text{if } \tau < T. \end{cases}$$

Now, taking the expectation of $\underline{V}_0(T)$ with respect to the random variable τ and assuming mutual independence between the deterioration and failure processes, we get

$$(24) \qquad \begin{aligned} \underline{V}_F(T) &= E_\tau\{\underline{V}_0(T)\} \\ &= \int_0^T \underline{V}(t)p_\tau(t;u)dt + \int_T^\infty \underline{V}(T)p_\tau(t;u)dt \\ &= \int_0^T \underline{V}(t)p_\tau(t;u)dt + Q_\tau(T;u)\underline{V}(T). \end{aligned}$$

Substituting (22) in (24) we get first

$$(25) \qquad \begin{aligned} \underline{V}_F(T) &= \int_0^T p_\tau(t;u)\underline{S}(t)\exp(-rt)dt + Q_\tau(T;u)\underline{S}(T)\exp(-rT) \\ &+ \int_0^T \{p_\tau(t;u)\int_0^t [\underline{p}(s)\underline{S}(s) - u(s)]\exp(-rs)ds\}dt \\ &+ Q_\tau(T;u)\int_0^T [\underline{p}(t)\underline{S}(t) - u(t)]\exp(-rt)dt, \end{aligned}$$

which after integrating by parts in the second integral and after simplificating (see Appendix) becomes

$$(26)\qquad \underline{V}_F(T) = Q_\tau(T;u)\underline{S}(T)\exp(-rT) + \int_0^T [\underline{p}_1(t)\underline{S}(t) - Q_\tau(t;u)u(t)]\exp(-rt)dt,$$

where we have denoted

$$(27)\qquad \underline{p}_1(t) = \underline{p}(t)Q_\tau(t;u) + p_\tau(t;u).$$

Our problem is now to choose an optimal maintenance policy $u^*(t)$ and an optimal sale date T so as to satisfy the state equations (19) and (20) and the control constraint (21) and to maximize the expectation

$$(28)\qquad \bar{V}_F(T) = E_P\{\underline{V}_F(T)\} = \int_\Omega \underline{V}_F(T)dP$$

where $\underline{V}_F(T)$ is given by (26) and the expectation is taken with respect to the probability measure P over the sample space Ω.

We can readily see that our generalized model is of the same form as the model Alam I, cf. equations (9) to (13), only with the coefficients of $\underline{S}(t)$ and u(t) modified. The generalized model coincides with the model Alam I, when we only set $p_\tau(t;u) \equiv 0$ (the failure part of the model is omitted). We can also see that our generalized model coincides with the model Alam II, if we in (26) set $T = \infty$ to give (17) (the sale date optimization is omitted). Our model thus contains both the models Alam I and Alam II as its special cases.

4 Solution by stochastic maximum principle

The solution of the problem formulated above needs an application of the stochastic maximum principle. For this we must assume certain smoothness and regularity conditions: f, β, and u are piecewise continuous, $\underline{\alpha}(t)$ and $\underline{p}(t)$ and, hence, $\underline{\delta}(t)$, $\underline{S}(t)$ and $\underline{V}_F(t)$ are stochastic processes and $\underline{\delta}_0$ a random variable on a sample space Ω

which is assumed to be a compact subset of an Euclidean space. The stochastic processes and the random variable are random quantities with respect to the probability measure P on Ω (for a detailed and strict description of the assumptions for the stochastic maximum principle see [4], pp. 876-878).

To solve the problem, we first form the Hamiltonian, which is now a random variable

$$(29)\qquad \underline{H} = H(\underline{S}, \underline{\delta}, u, \underline{\lambda}_1, \underline{\lambda}_2, t) = -\{[\underline{p}(t)Q_\tau(t;u) + p_\tau(t;u)]\underline{S}(t) - Q_\tau(t;u)u(t)\}\exp(-rt) + \underline{\lambda}_1(t)[-\underline{\delta}(t) + f(t)u(t)] + \underline{\lambda}_2(t)[\underline{\alpha}(t) - \beta(t)u(t)],$$

where the adjoint variables $\underline{\lambda}_1(t)$ and $\underline{\lambda}_2(t)$ also are stochastic processes on Ω and satisfy the stochastic differential equations

$$(30)\qquad \frac{d\underline{\lambda}_1(t)}{dt} = -\frac{\partial \underline{H}}{\partial \underline{S}} = [\underline{p}(t)Q_\tau(t;u) + p_\tau(t;u)]\exp(-rt)$$

and

$$(31)\qquad \frac{d\underline{\lambda}_2(t)}{dt} = -\frac{\partial \underline{H}}{\partial \underline{\delta}} = \underline{\lambda}_1(t)$$

with the boundary conditions

$$(32)\qquad \underline{\lambda}_1(T) = -\frac{\partial}{\partial \underline{S}}[Q_\tau(T;u)\underline{S}(T)\exp(-rT)] = -Q_\tau(T;u)\exp(-rT)$$

and

$$(33)\qquad \underline{\lambda}_2(T) = -\frac{\partial}{\partial \underline{\delta}}[Q_\tau(T;u)\underline{S}(T)\exp(-rT)] = 0.$$

To find the solution for our problem we should proceed as follows. First we consider T as fixed and apply the stochastic maximum principle (i.e. minimize $E_p\{\underline{H}\}$ with respect to u, see [4] pp. 879-880) to obtain the optimal maintenance policy $u^*(t)$ for $0 \le t \le T$. Then we choose T so as to maximize $\bar{V}_F(T)$. This is done by differentiating $\bar{V}_F(T)$ with respect to T and setting it equal to zero.

There exist, however, two reasons, why an analytic solution for this general case is not possible, and, in order to find out the solution, we had to use one of the interative computational techniques. First, equations (30) and (31) are general stochastic differential equations, and secondly, the failure probability $p_\tau(t;u)$ depends on the maintenance performed. Here we present the solution for the problem in the special case where an analytic solution is possible to obtain. Therefore, we make the following additional assumptions. First we assume that $\underline{\alpha}(t)$ and $\underline{p}(t)$ are independent of t, $\underline{\alpha}(t) \equiv \underline{\alpha}$ and $\underline{p}(t) \equiv \underline{p}$ are simply random variables. The assumption makes it possible to obtain an explicit solution for the co-state equations (30) to (33) and, hence, for the state equations (19) and (20). The solution is achieved by replacing the required stochastic quantities with their expected values ([8], p. 416). The second assumption is that the failure probability is independent of maintenance: $p_\tau(t;u) = p_\tau(t)$ and, hence, $Q_\tau(t;u) = Q_\tau(t)$. With this assumption, an analytic application of the stochastic maximum principle is possible.

5 Conditions for optimal maintenance policy and sale date

The Hamiltonian for this special optimal control problem becomes, cf. the general case (29),

(34) $$\underline{H} = H(\underline{S}, \underline{\delta}, u, \underline{\lambda}_1, \underline{\lambda}_2, t)$$
$$= - \{[\underline{p}Q_\tau(t) + p_\tau(t)]\underline{S}(t) - Q_\tau(t)u(t)\}\exp(-rt)$$
$$+ \underline{\lambda}_1(t)[-\underline{\delta}(t) + f(t)u(t)] + \underline{\lambda}_2(t)[\underline{\alpha} - \beta(t)u(t)],$$

where the adjoint variables $\underline{\lambda}_1(t)$ and $\underline{\lambda}_2(t)$ are now given by

(35) $$\begin{cases} \dfrac{d\underline{\lambda}_1(t)}{dt} = [\underline{p}Q_\tau(t) + p_\tau(t)]\exp(-rt) \\ \underline{\lambda}_1(T) = - Q_\tau(T)\exp(-rT) \end{cases}$$

and

(36) $$\begin{cases} \dfrac{d\underline{\lambda}_2(t)}{dt} = \underline{\lambda}_1(t) \\ \underline{\lambda}_2(T) = 0. \end{cases}$$

Applying the stochastic maximum principle, i.e. minimizing $E_p\{\underline{H}\}$ with respect to u, the following condition for the optimal maintenance policy $u^*(t)$ is obtained

(37) $$u^*(t) = \begin{cases} U, \text{ if } E_p\{G(Q_\tau, \underline{\lambda}_1, \underline{\lambda}_2, t)\} < 0 \\ \text{arbitrary} \in [0,U], \text{ if } E_p\{G(Q_\tau, \underline{\lambda}_1, \underline{\lambda}_2, t)\} = 0 \\ 0, \text{ if } E_p\{G(Q_\tau, \underline{\lambda}_1, \underline{\lambda}_2, t)\} > 0. \end{cases}$$

In (37) we have denoted

(38) $$G(Q_\tau, \underline{\lambda}_1, \underline{\lambda}_2, t) = Q_\tau(t)\exp(-rt) + \underline{\lambda}_1(t)f(t) - \underline{\lambda}_2(t)\beta(t).$$

Equation (37) shows that the optimal maintenance policy, under the assumptions made, is still bang-bang. The possible switching point(s) T', where the level of maintenance is changed from U to 0 or vice versa, satisfy the switching equation

(39) $$E_p\{G(Q_\tau, \underline{\lambda}_1, \underline{\lambda}_2, T')\}$$
$$= Q_\tau(T')\exp(-rT') + f(T')E_p\{\underline{\lambda}_1(T')\} - \beta(T')E_p\{\underline{\lambda}_2(T')\}$$
$$= Q_\tau(T')\exp(-rT') + f(T')\bar{\lambda}_1(T') - \beta(T')\bar{\lambda}_2(T') = 0$$

or

$$(40) \qquad f(T') = \frac{\beta(T')\bar{\lambda}_2(T') - Q_\tau(T')\exp(-rT')}{\bar{\lambda}_1(T')}$$

($\bar{\lambda}_1(t)$ and $\bar{\lambda}_2(t)$ denote the expectations of $\underset{\sim}{\lambda}_1(t)$ and $\underset{\sim}{\lambda}_2(t)$, respectively, the expectations being with respect to the probability measure P and conditional on t).

Thus far we have considered the planned sale date T as fixed. We still have to choose T so as to maximize the expected present value $\bar{V}_F(T)$ given by equation (28) (in the expression of $\bar{V}_F(T)$ we now use, of course, the optimal maintenance policy u^* as the control variable u, and take the assumptions of this section also into account). To maximize $\bar{V}_F(T)$, we differentiate (28) with respect to T and set the derivative equal to zero. Using similar reasoning as Thompson ([10], p. 546), and assuming mutual independence between the random variables $\underset{\sim}{\alpha}$ and $\underset{\sim}{p}$ we get first

$$(41) \qquad \frac{d\bar{V}_F(T)}{dT} = \frac{dQ_\tau(T)}{dT}\bar{S}(T)\exp(-rT) + Q_\tau(T)\frac{d\bar{S}(T)}{dT}\exp(-rT)$$
$$- rQ_\tau(T)\bar{S}(T)\exp(-rT)$$
$$+ [\bar{p}_1\bar{S}(T) - Q_\tau(T)u^*(T)]\exp(-rT),$$

which after substitution of equations (19) and (27) and rearrangement of terms becomes

$$(42) \qquad \frac{d\bar{V}_F(T)}{dT} = Q_\tau(T)\exp(-rT)\{(\bar{p}-r)\bar{S}(T) - \bar{\delta}(T) + [f(T) - 1]u^*(T)\}.$$

The optimal planned sale date T is thus reached when

$$(43) \qquad \bar{S}(T) = \{\bar{\delta}(T) - [f(T)-1]u^*(T)\}/(\bar{p}-r).$$

In equations (41) to (43) the symbol ($\bar{\ }$) again denotes the expected value of the random variable, the expectation being with respect

to the probability measure P. From (43) we can see that the optimal planned sale date is independent of the life time distribution.

6 A particular case: exponentially distributed life time

Equations (37) and (43) give the general conditions for the optimal maintenance policy and for the optimal sale date. We shall now demonstrate the explicit calculation of the optimal maintenance policy (37) for a machine with exponentially distributed life time. Therefore, let $p_\tau(t) = \sigma\exp(-\sigma t)$ and, hence, $Q_\tau(t) = \exp(-\sigma t)$ (for $t \geq 0$). As is well known, the parameter of the distribution ($=\sigma$) corresponds to the (constant) failure rate of the machine.

Substituting the above expressions for $p_\tau(t)$ and $Q_\tau(t)$ into (35) and (36) and integrating, we get the expected values of the adjoint variables $\underset{\sim}{\lambda}_1(t)$ and $\underset{\sim}{\lambda}_2(t)$ as

$$(44) \qquad \bar{\lambda}_1(t) = -\exp\{-(r+\sigma)t\}[(\bar{p}+\sigma) - (\bar{p}-r)\exp\{-(r+\sigma)(T-t)\}]/(r+\sigma)$$

and

$$(45) \qquad \bar{\lambda}_2(t) = (\bar{p}+\sigma)\exp\{-(r+\sigma)t\}[1 - \exp\{-(r+\sigma)(T-t)\}]/(r+\sigma)^2$$
$$- [(\bar{p}-r)/(r+\sigma)]\exp\{-(r+\sigma)T\}(T-t).$$

Taking the expectation of (38) and substituting (44) and (45) into it, we get

$$(46) \qquad \bar{G}(t) = E_P\{G(Q_\tau,\underset{\sim}{\lambda}_1,\underset{\sim}{\lambda}_2,t)\} = G(Q_\tau,\bar{\lambda}_1,\bar{\lambda}_2,t)$$
$$= \exp\{-(r+\sigma)t\}\Big\{(r+\sigma) - f(t)\Big[(\bar{p}+\sigma) - (\bar{p}-r)\exp\{-(r+\sigma)(T-t)\}\Big]$$
$$- \beta(t)\Big[(\bar{p}+\sigma)[1 - \exp\{-(r+\sigma)(T-t)\}]/(r+\sigma)$$
$$- (\bar{p}-r)\exp\{-(r+\sigma)(T-t)\}(T-t)\Big]\Big\}/(r+\sigma).$$

The optimal maintenance policy (37) thus becomes

$$u^*(t) = \begin{cases} U, \text{ if } \bar{G}(t) < 0 \\ \text{arbitrary} \in [0,U], \text{ if } \bar{G}(t) = 0 \\ 0, \text{ if } \bar{G}(t) > 0. \end{cases} \quad (47)$$

The bang-bang optimal policy (47) has none, one or more switching points. For a switching point T' we have $\bar{G}(T') = 0$. The possible switching points may be computed using (46) or (40) with (44) and (45).

In section 3 we showed that our generalized model contains the prior models Alam I and Alam II as its special cases. As we now in the exponential case have obtained an explicit solution for the problem, we can also compare the results. At the same time we may obtain an interesting economic interpretation for the failure rate parameter σ.

Comparing the optimal maintenance policy (47) with the optimal policy (14) in the model Alam I, we see that they are of exactly the same form. If we in (14) instead of the discount rate r use the 'risk-adjusted' discount rate $r+\sigma$ and instead of the mean production rate $\bar{p}$ use the 'risk-adjusted' mean production rate $\bar{p}+\sigma$, we get (47). Or on the contrary, if we in our model ignore the possibility of random failure and set $\sigma = 0$, (47) gives (14). This leads to the following economic interpretation for the parameter σ. The failure rate may be interpreted as a risk premium which is used to adjust both the discount rate and the mean production rate to the level of those in a certainty-equivalent problem. The parameter σ is a tool with the help of which the degree of uncertainty caused by the random life of the machine can be measured and expressed in money terms.

In the model Alam II, instead of optimizing also the sale date of the machine, the machine was assumed to be kept until it fails and becomes junked, or in our terms, the planned sale date was fixed to infinity. Setting $T = \infty$ in (46) we see indeed, that (47) coincides with (18), the optimal maintenance policy for the model Alam II is obtained as a special case of our optimal policy (47).

7 An example

To compare the three different models (Alam I, Alam II, our generalized model) also numerically, we shall now consider a simple example in which the following specific values are assumed for various quantities:

$S_0 = 100$	(FIM 1000)
$U = 1$	(FIM 1000 per year)
$\bar{\delta}_0 = 2$	(FIM 1000 per year)
$\bar{p} = 0.10$	(per year)
$r = 0.05$	(per year)
$\bar{\alpha} = 0.05$	(FIM 1000 per year square)
$\sigma = 0.10$	(per year)
$\beta(t) \equiv 0.01$	(per year; constant)
$f(t) = 2\exp(-0.10\,t)$	
time t is in years	

First we derive the time paths of the state variables $\underline{\delta}$ and $\underline{S}$ as a solution for the state equations (19) and (20) (these state equations also hold for the other models Alam I and Alam II). The optimal maintenance policy u^* is, as it will later on turn out, in all the three cases of the following type:

$$u^*(t) = \begin{cases} U, \text{ when } t \in [0,T') \\ \text{arbitrary} \in [0,U], \text{ when } t = T' \\ 0, \text{ when } t \in (T',T]. \end{cases} \quad (48)$$

In (48), the maintenance switch-off time T' and the sale date T vary, of course, from one model to another (in the model Alam II we have the fixed $T = \infty$). As was stated before (see p. 12), the solution of the state equations is achieved by replacing all the stochastic quantities with their expected values. We get

$$\bar{\delta}(t) = \begin{cases} 2 + 0.04\,t \text{ for } t \in [0,T'] \\ 2 - 0.01\,T' + 0.05\,t \text{ for } t \in (T',T] \end{cases} \quad (49)$$

and

$$(50)\qquad \bar{S}(t) = \begin{cases} 120 - 2t - 0.02t^2 - 20\exp(-0.10t) & \text{for } t \in [0,T'] \\ 120 - 0.005T'^2 - 20\exp(-0.10T') - (2 - 0.01T')t \\ - 0.025t^2 \quad \text{for } t \in (T',T]. \end{cases}$$

For a machine with random life time (the model Alam II, the generalized model) the time paths (49) and (50) are relevant until the possible failure only. Now we consider the different models one at a time.

The model Alam I. From equations (14) and (15), the optimal maintenance policy may be computed and is given by

$$(51)\qquad u^*(t) = \begin{cases} \text{FIM 1000 per year} & \text{for } 0 \le t \le 5.0 \\ \text{FIM 0 per year} & \text{for } 5.0 < t \le 19.9 \end{cases}$$

so that the time of the optimal maintenance switch-off is $T' = 5.0$ years and the optimal sale date is $T = 19.9$ years. The expected salvage value of the machine is $\bar{S}(T)$ = FIM 58 900 and the maximum of the expected present value of the machine is $\bar{V}(T)$ = FIM 129 900. The expected net profit from the use of the machine thus becomes $\bar{V}(T) - S_o$ = FIM 29 900.

The model Alam II. From equation (18) the following optimal maintenance policy is obtained:

$$(52)\qquad u^*(t) = \begin{cases} \text{FIM 1000 per year} & \text{for } 0 \le t \le 10.7 \\ \text{FIM 0 per year} & \text{for } t > 10.7. \end{cases}$$

The time of the optimal maintenance switch-off is now $T' = 10.7$ years (and the selling of the machine is not considered, the machine is used until it fails and becomes junked). We see that the switching from high maintenance to no maintenance takes place later when random catastrophic failure is taken, but the option of selling a still operable machine is not taken into account. We may now be forced to keep the machine for a long time, so we also upkeep maintenance for a longer period. The maximal expected present value of the machine is now $\bar{V}_F$ = FIM 104 800, so that the expected net profit becomes only $\bar{V}_F - S_o$ = FIM 4 800.

The generalized model. Using equations (43) and (47) together with (46) we may compute $T' = 10.0$ years and $T = 21.6$ years. The expected salvage value of the machine (if not failed) is $\bar{S}(T)$ = FIM 58500 and the optimal expected present value is $\bar{V}_F(T)$ = FIM 112000. The expected net profit is now $\bar{V}_F(T) - S_o$ = FIM 12000, which is 2.5 times as much as in that case where the sale date optimization was omitted (Alam II). Thus we see the importance of the sale date optimization also in the case of stochastic machine life. Comparing the case without failure (Alam I) to the general case with random failure taken into account we notice that the expected net profit has decreased from FIM 29900 to FIM 12000. This is, of course, due to the possibility of machine failing before the optimal sale date has been reached.

8 Conclusion

We have examined the problem of optimal maintenance and optimal sale date of a machine when it is subject to random deterioration and random catastrophic failure, generalizing and unifying the recent prior work on the problem. Especially we have pointed out the importance of the sale date optimization also in the case of random machine life. A particular case with an example has been presented to show the effect of this optimization as well as the effect of random failure on the results obtained.

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Appendix

We write (25) into the form

(A1) $$\underline{V}_F(T) = Q_\tau(T;u)\underline{S}(T)\exp(-rT) + \int_0^T p_\tau(t;u)\underline{S}(t)\exp(-rt)dt + Q_\tau(T;u)\int_0^T[\underline{p}(t)\underline{S}(t) - u(t)]\exp(-rt)dt + \int_0^T p_\tau(t;u)\{\int_0^t[\underline{p}(s)\underline{S}(s) - u(s)]\exp(-rs)ds\}dt$$

and denote

(A2) $$I = \int_0^T p_\tau(t;u)\{\int_0^t[\underline{p}(s)\underline{S}(s) - u(s)]\exp(-rs)ds\}dt.$$

Then we can write

(A3) $$I = \int_0^T - dQ_\tau(t;u)\{\int_0^t[\underline{p}(s)\underline{S}(s) - u(s)]\exp(-rs)ds\} = -\int_0^T dQ_\tau(t;u)v(t)$$

where

(A4) $$v(t) = \int_0^t[\underline{p}(s)\underline{S}(s) - u(s)]\exp(-rs)ds.$$

Integrating I by parts, we get

(A5) $$I = \Big/_0^T - Q_\tau(t;u)v(t) + \int_0^T Q_\tau(t;u)d\,v(t) = -Q_\tau(T;u)\int_0^T[\underline{p}(t)\underline{S}(t) - u(t)]\exp(-rt)dt + \int_0^T Q_\tau(t;u)[\underline{p}(t)\underline{S}(t) - u(t)]\exp(-rt)dt.$$

Substituting (A5) into (A1), we get

(A6) $$\underline{V}_F(T) = Q_\tau(T;u)\underline{S}(T)\exp(-rT) + \int_0^T p_\tau(t;u)\underline{S}(t)\exp(-rt)dt + \int_0^T Q_\tau(t;u)[\underline{p}(t)\underline{S}(t) - u(t)]\exp(-rt)dt$$
$$= Q_\tau(T;u)\underline{S}(T)\exp(-rT) + \int_0^T\{[p_\tau(t;u) + Q_\tau(t;u)\underline{p}(t)]\underline{S}(t) - Q_\tau(t;u)u(t)\}\exp(-rt)dt,$$

which coincides with the desired result (26).

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